Qualifying Exam Syllabus

Seewoo Lee

Exam Date: April 26 (Friday), 2019
Time: 1pm – 4pm
Location: Evans 748

Committee: SugWoo Shin (Advisor), Paul Vojta (Chair), David Nadler, Ori Ganor (Outside member).

Syllabus

1 Major topic: Automorphic Forms (Algebra)

References: Bump, *Automorphic Forms and Representations*, Chapter 2.1 - 2.8, 3.1 - 3.6, 4.2 - 4.8

- Tate thesis (GL(1) theory): local and global functional equations of the ζ and L-functions, analytic continuation of the L-function.
- Archimedean local theory ($\mathrm{GL}(2,\mathbb{R})$ theory): Basic Lie theory, universal enveloping algebras, Laplacian, $(\mathfrak{g}, K)$-module, classification of irreducible admissible $(\mathfrak{g}, K)$-module of $\mathrm{GL}(2,\mathbb{R})$, construction of principal and discrete series representations, unitaricity, Whittaker models,
- non-archimedean local theory: Admissible and smooth representations, non-Archimedean Hecke algebra, construction of principal series, supercuspidal representations, Jacquet functor, Whittaker model and Kirillov model
- Global theory: adelization of classical forms, tensor product theorem (statement), multiplicity one theorem (statement).

2 Major topic: Number Theory (Algebra)

References: Neukirch, *Algebraic Number Theory*, Chapter I (1 - 10), II (1 - 10)
Milne, *Class Field Theory*, Chapter II, III, V, VI (but no proofs - only statements)

- Number Fields: Integrality, norm and trace, Dedekind domains, factorization of ideals, ideal class groups and finiteness thereof, Minkowski bound, Dirichlet's unit theorem, discriminant, Kummer's theorem on explicit factorizations of ideals, factorization in Galois extensions (Hilbert theory);
- Valuations: p-adic numbers; completions, valuations and absolute values, extensions of valuations, Hensel's lemma, unramified, ramified and tamely ramified extensions; ramfication groups
- Class Field Theory: local and global class field theory (statement); Artin reciprocity (statement), Chebotarev density theorem (statement).

3 Minor topic: Complex Analysis (Analysis)

References: Stein and Shakarchi, *Complex Analysis*, Chapter 1 - 3, 6 - 7, 9 - 10

- Basic complex analysis: Cauchy-Riemann equation, Cauchy integral theorem, Cauchy integral formula, open mapping theorem, maximum principle, residue theorem, argument principle
- Additional topics: gamma function, zeta function, prime number theorem (sketch of the proof), elliptic functions, Jacobi theta function and sum of squares

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Questions

4 Automorphic forms

- Explain Tate thesis. (Sugwoo)
  - What is a Dirichlet L-function? What is an analytic continuation and functional equation? (Sugwoo)
  - How to Adélize Dirichlet character? (Sugwoo)
  - Where do we use Fourier theory on Adeles? (Sugwoo)
  - How this can be used to prove functional equation and analytic continuation of Dirichlet L-function? (Sugwoo)
- What are the representations of $\mathrm{GL}(1, \mathbb{R})$? (Nadler)
- State a classification of $(\mathfrak{g}, K)$-modules of $\mathrm{GL}(2, \mathbb{R})$. (Nadler)
  - What is a relation between $\mathfrak{gl}(2, \mathbb{C})$ and $\mathfrak{sl}(2, \mathbb{C})$? (Nadler)
  - What is a difference between $(\mathfrak{g}, K)$-modules of $\mathrm{GL}(2, \mathbb{R})_+$ and $\mathrm{GL}(2, \mathbb{R})$? (Sugwoo)

5 Number theory

- Find all the quadratic extensions of $\mathbb{Q}_3$ (Vojta)
  - Describe structures of $\mathbb{Q}_3^*$ and $\mathbb{Q}_3^*/(\mathbb{Q}_3^*)^2$. (Sugwoo)
- How the prime 3 can be factorized in cubic number fields? Give a possible list. (Vojta)
  - Give an example of non-Galois cubic extension example that 3 factorizes as two primes with different ramification index or different inertia degree. (Sugwoo)
- State the Hensel's lemma (as in Neukirch) (Vojta)
  - What are the conditions on the reduced polynomials $\bar{g}, \bar{h}$? Give an example that the factorization does not lift. (Sugwoo)

6 Complex analysis

- Find all the holomorphic maps from $\mathbb{CP}^1$ to $\mathbb{CP}^1$. (Nadler)
  - How to count number of poles of a given holomorphic function? (Ganor)
- What are the holomorphic maps from $\mathbb{CP}^1$ to complex torus or torus to $\mathbb{CP}^1$? (Nadler)
  - What are the zeros and poles of Weierstrass function? (Nadler)
  - What are the simply connected Riemann surfaces? (Nadler)
  - Why the map $\mathbb{CP}^1 \to \mathbb{C}/\Lambda$ can be lifted to a map $\mathbb{CP}^1 \to \mathbb{C}$? What do you get from this? (Nadler)