Algebraic proof of modular form inequalities for optimal sphere packings

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Goal

- We develop simple but surprisingly useful tools to study *(completely) positive quasimodular forms*.
- Using the theory, we give *algebraic* proofs of Viazovska and Cohn–Miller–Kumar–Radchenko–Viazovska's modular form inequalities for the E_8 and Leech lattice packing in dimensions 8 and 24.
- We also prove a conjecture of Kaneko and Koike for the extremal forms in the case of depth 1.

Sphere packing

Question

For given $d \geq 1$*, find an optimal sphere (in fact, ball) packing of* $\mathbb{R}^d$ *and its density* Δ_d*.*

Sphere packing

Before 2016:

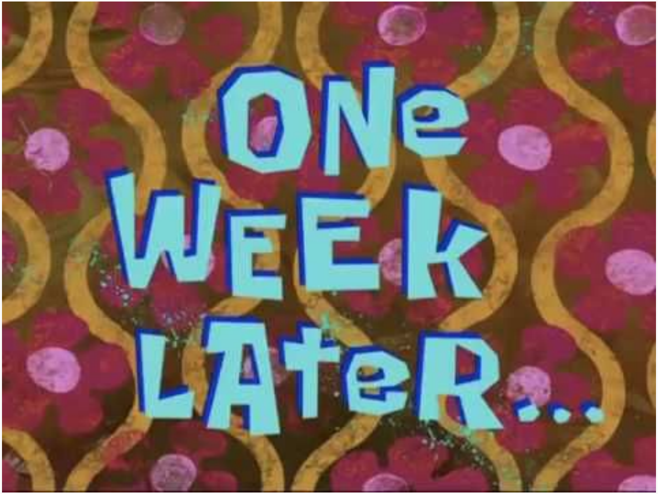
- $d = 1$. $\Delta_1 = 1$.
- $d = 2$. (Thue 1910) $\Delta_2 = \frac{\pi}{2\sqrt{3}}$ (hexagonal (A_2) packing).
- $d = 3$. (Kepler conjecture 1611, Hales 2005) $\Delta_3 = \frac{\pi}{3\sqrt{2}}$ (Cannon ball packing). Tons of computer calculations, formally verified in 2014 using Isabelle/HOL light.
- (Korkine–Zolotareff, Blichfeldt, Cohn–Kumar) D_4, D_5, D_6, E_7, E_8, and Leech lattice packings are optimal among *lattice* packings for $d = 4, 5, 6, 7, 8, 24$ respectively.

Sphere packing, $d = 8$

Theorem (Viazovska, 2016 π-day on arXiv)

E_8 lattice packing is optimal with $\Delta_8 = \frac{\pi^4}{384}$.

$$E_8 = \left\{ (x_i) \in \mathbb{Z}^8 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^8 : \sum_{i=1}^{8} x_i \equiv 0 \pmod{2} \right\} \subset \mathbb{R}^8$$


ONE
WEEK
LATER...

Sphere packing, $d = 24$

Theorem (Cohn–Kumar–Miller–Radchenko–Viazovska, March 21st 2016 on arXiv)

Leech lattice packing is optimal with $\Delta_{24} = \frac{\pi^{12}}{12!}$.

Unique even unimodular lattice with nonzero minimial length $\lambda(\Lambda_{24}) = 2$. Can be constructed by the binary Golay code, Lorentzian lattice $II_{25,1}$, etc.

LP bound

How? We have a **Linear programming bound** for sphere packing:

Theorem (Cohn–Elkies, 2003)

Let $r > 0$. Assume that there exists a nice function $f : \mathbb{R}^d \to \mathbb{R}$ satisfying

- $f(0) = \widehat{f}(0) > 0$,
- $f(x) \leq 0$ *for all* $\|x\| \geq \mathbf{r}$,
- $\widehat{f}(y) \geq 0$ *for all* $y \in \mathbb{R}^d$.

Then

$$\Delta_d \leq \mathrm{vol}(B^d_{r/2}) = \left(\frac{r}{2}\right)^d \frac{\pi^{d/2}}{(d/2)!}.$$

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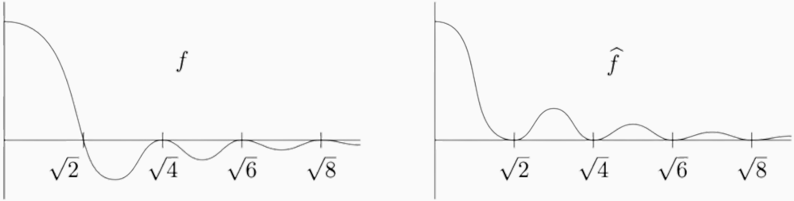
It may look like this (for $d = 8$, radial):



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Decompose f into Fourier eigenfunctions $f = f_+ + f_-$, where $\widehat{f_+} = f_+$ and $\widehat{f_-} = -f_-$. Viazovska write them as

$$f_\pm(x) = \sin^2\left(\frac{\pi\|x\|^2}{2}\right)\int_0^\infty \varphi_\pm(t)e^{-\pi\|x\|^2 t}\mathrm{d}t,$$

where $\sin^2$ factor is included to enforce desired roots. Then $f_\pm$ being Fourier eigenfunctions correspond to $\varphi_\pm$ being "(quasi)modular forms".

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where $\sin^2$ factor is included to enforce desired roots. Then $f_\pm$ being Fourier eigenfunctions correspond to $\varphi_\pm$ being "(quasi)modular forms". Now the linear constraints (inequalities) on f and $\widehat{f}$ reduces to the modular inequalities

$$\varphi_+(t) + \varphi_-(t) < 0,$$
$$\varphi_+(t) - \varphi_-(t) > 0.$$

$d = 8$, modular form inequalities

For $d = 8$, $\varphi_{\pm}(t) = t^2\psi_{\pm}(i/t)$, $\psi_+ = -\frac{F}{\Delta}$, $\psi_- = -\frac{18}{\pi^2}\frac{G}{\Delta}$ where

$$F = (E_2E_4 - E_6)^2$$
$$G = H_2^3(2H_2^2 + 5H_2H_4 + 5H_4^2),$$

and $H_2 = \Theta_2^4$ and $H_4 = \Theta_4^4$. Then the inequality reduces to

$$F(it) + \frac{18}{\pi^2}G(it) > 0,$$
$$F(it) - \frac{18}{\pi^2}G(it) < 0.$$

$d = 24$, modular form inequalities

For $d = 24$ we have the following *three* inequalities:

$$F(it) + \frac{432}{\pi^2} G(it) \geq 0,$$
$$F(it) - \frac{432}{\pi^2} G(it) \leq 0.$$
$$t^{10} \left(-\frac{F(i/t)}{\Delta(i/t)^2} + \frac{432}{\pi^2} \frac{G(i/t)}{\Delta(i/t)^2} \right) \geq \frac{725760}{\pi} e^{2\pi t} \left(t - \frac{10}{3\pi} \right)$$

($t \geq 1$ enough for the last one) where

$$F = 49E_2^2E_4^3 - 25E_2^2E_6^2 - 48E_2E_4^2E_6 - 25E_4^4 + 49E_4E_6^2,$$
$$G = H_2^5(2H_2^2 + 7H_2H_4 + 7H_4^2).$$

Some observations on the inequalities

For $d = 8$, the first inequality is "easy": we have $F(it) > 0$ and $G(it) > 0$ for all $t > 0$. But the second inequality is "hard": we need to compare modular forms of different weights (12 and 10).

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For $d = 8$, the first inequality is "easy": we have $F(it) > 0$ and $G(it) > 0$ for all $t > 0$. But the second inequality is "hard": we need to compare modular forms of different weights (12 and 10).

For $d = 24$, even the first inequality is not so trivial: $F(it) > 0$ is not obvious. Second one is hard, and the last one is harder.

Previous proofs

Viazovska and CKMRV's original proofs use bounds of Fourier coefficients of the form

$$|c(n)| \leq C_1 e^{C_2 \pi \sqrt{n}}$$

(follows from the Hardy–Ramanujan formula) and reduce inequalities to finite calculations + interval arithmetic.

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Dan Romik (2023) gave an alternative and much simpler proof *for* $d = 8$ that does not use any of interval arithmetic. But it still requires a "calculator" to check inequalities like

$$e^{3\pi} \frac{9\Gamma(1/4)^{16}}{8192\pi^{12}} < 20480.$$

Also, $0 < t < 1$ and $t \geq 1$ cases are treated separately.

Algebraic proof

Question

Can we prove these **algebraically***?*

(Completely) positive quasimodular forms

To (re-)prove these modular form inequalities, we develop some theory of **(completely) positive quasimodular forms**.

(Completely) positive quasimodular forms

Definition

Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. We call $F \in \mathcal{QM}_w^s(\Gamma)$ a **positive quasimodular form** if

$$F(it) \geq 0$$

for all $t > 0$. We denote $\mathcal{QM}_w^{s,+}(\Gamma)$ for the set of positive quasimodular forms.

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We call $F \in \mathcal{QM}_w^s(\Gamma)$ a **completely positive quasimodular form** if it has nonnegative q-coefficients at ∞. We denote $\mathcal{QM}_w^{s,++}(\Gamma)$ for the set of completely positive quasimodular forms.

(Completely) positive quasimodular forms

We have $\mathcal{QM}_w^{s,++} \subseteq \mathcal{QM}_w^{s,+} \subseteq \mathcal{QM}_w^{s}$, and the two sets form a convex cone in $\mathcal{QM}_w^{s}$.

The inclusion is strict in general:
$\Delta(q) = q\prod_{n\geq 1}(1-q^n)^{24} = q - 24q^2 + \cdots$ is positive but not completely positive.

Positive forms and derivatives

"Easy" facts, one-line proofs:

Theorem (L.)

1. *Anti-derivative preserves positivity.*
2. *Derivative preserves complete positivity.*
3. *Serre derivative preserves complete positivity.*

"Nontrivial" fact[1] (that we won't use):

Theorem (L.)

F is completely positive if and only if all its derivatives are positive.

[1] almost directly follows from Bernstein's theorem

Positive forms and Serre derivatives

"Interesting" fact that we will use:

Theorem (L.)

Let $F = \sum_{n \geq n_0} a_n q^n \in \mathcal{QM}_w^s$ *be a quasimodular form of real coefficients with* $a_{n_0} > 0$. *If* $\partial_k F \in \mathcal{QM}_{w+2}^{s+1,+}$ *for some* k, *then* $F \in \mathcal{QM}_w^{s,+}$.

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In other words, anti-Serre-derivative preserves positivity.

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{F(it)}{\Delta(it)^{k/12}}\right) = (-2\pi)\frac{(\partial_k F)(it)}{\Delta(it)^{k/6}} < 0.$$

□

Extremal forms

Examples?

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Definition (Kaneko–Koike)

For a given weight w and depth s, **extremal quasimodular form of weight w and depth s**, $X_{w,s}$, is a quasimodular form of *largest possible vanishing order at the cusp*. More precisely, $X_{w,s}$ admits a q-expansion

$$X_{w,s} = \sum_{n \geq m} a_n q^n$$

where $m = \dim_{\mathbb{C}} \mathcal{QM}_w^s - 1$ and $a_m \neq 0$.

Examples

$$X_{6,1} = \frac{E_2 E_4 - E_6}{720} = q + 18q^2 + 84q^3 + 292q^4 + 630q^5 + \cdots$$

$$X_{8,1} = \frac{-E_2 E_6 + E_4^2}{1008} = q + 66q^2 + 732q^3 + 4228q^4 + 15630q^5 + \cdots$$

$$X_{4,2} = \frac{-E_2^2 + E_4}{288} = q + 6q^2 + 12q^3 + 28q^4 + 30q^5 + \cdots$$

$$X_{8,2} = \frac{-7E_2^2 E_4 + 2E_2 E_6 + 5E_4^2}{362880} = q^2 + 16q^3 + 102q^4 + 416q^5 + \cdots$$

$$X_{6,3} = \frac{5E_2^3 - 3E_2 E_4 - 2E_6}{51840} = q^2 + 8q^3 + 30q^4 + 80q^5 + \cdots$$

Theorems and conjectures

For each $1 \leq s \leq 4$:

- **Theorem** (Pellarin) extremal forms of weight w and depth s exists and unique up to a constant.
- **Theorem** (Kaneko–Koike, Grabner) extremal forms satisfy recurrence relations and differential equations.
- **Conjecture** (Kaneko–Koike) extremal forms have nonnegative q-coefficients (i.e. completely positive).
    - **Theorem** (Grabner) Conjecture is true *for all but finitely many coefficients*.

Kaneko–Koike conjecture for $s = 1$

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Proof.

For $w \equiv 0 \pmod 6$ and $w \geq 12$, we can prove

$$\begin{aligned} X'_{w,1} &= \frac{5w}{72} X_{6,1} X_{w-4,1} + \frac{7w}{72} X_{8,1} X_{w-6,1} \\ X'_{w+2,1} &= \frac{5w}{72} X_{6,1} X_{w-2,1} + \frac{7w}{12} X_{8,1} X_{w-4,1} \\ X'_{w+4,1} &= 240 X_{6,1} X_{w,1} + \frac{7w}{72} X_{8,1} X_{w-2,1} + \frac{5w}{72} X_{10,1} X_{w-4,1} \end{aligned}$$

and these imply nonnegativity of q-coefficients. □

Kaneko–Koike conjecture for $s = 2$

We also have similar identities proving complete positivity of the depth 2 extremal forms of weight $w \leq 14$:

$$
\begin{aligned}
X_{4,2} &= \frac{1}{24}(-E_2') \\
X_{8,2}' &= 2X_{4,2}X_{6,1} \\
X_{10,2}' &= \frac{8}{9}X_{4,2}X_{8,1} + \frac{10}{9}X_{6,1}^2 \\
X_{12,2}' &= 3X_{6,1}X_{8,2} \\
X_{14,2}' &= 3X_{4,2}X_{12,1}
\end{aligned}
$$

but we don't have ones for general w yet.

$d = 8$, new proof

Let's get back to the modular form inequality for $d = 8$ case. The second (hard) inequality was

$$F(it) < \frac{18}{\pi^2} G(it)$$

where

$$\begin{aligned} F &= (E_2 E_4 - E_6)^2 \\ G &= H_2^3 (2H_2^2 + 5H_2 H_4 + 5H_4^2). \end{aligned}$$

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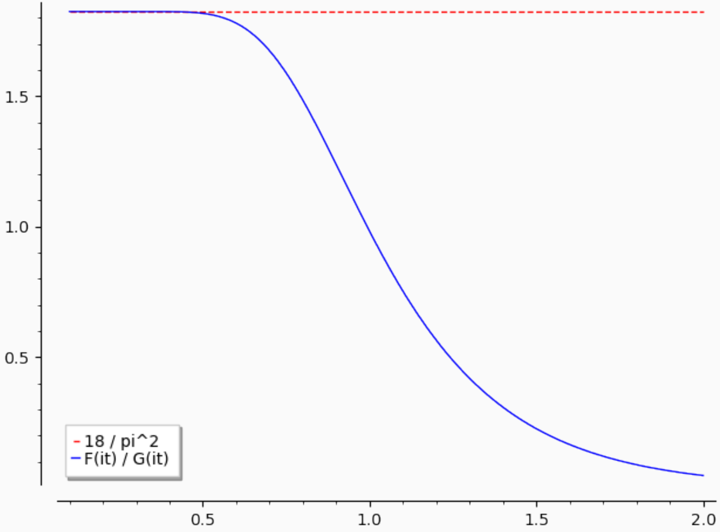
$d = 8$, homogenization



This graph tells us what we should try:

Proposition

$$\lim_{t \to 0^+} \frac{F(it)}{G(it)} = \frac{18}{\pi^2}.$$

Proposition

The function

$$t \mapsto \frac{F(it)}{G(it)}$$

is decreasing in t.

and both turned out to be true.

$d = 8$: limit

Proof of the limit.

We have

$$\lim_{t \to 0^+} \frac{F(it)}{G(it)} = \lim_{t \to \infty} \frac{F(i/t)}{G(i/t)}$$

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and F and G satisfy the following functional equations:

$$F\left(\frac{i}{t}\right) = t^{12} F(it) - \frac{12t^{11}}{\pi}(E_2(it)E_4(it) - E_6(it))E_4(it) + \frac{36t^{10}}{\pi^2} E_4(it)^2,$$

$$G\left(\frac{i}{t}\right) = t^{10} H_4(it)^3 (2H_4(it)^2 + 5H_4(it)H_2(it) + 5H_2(it)^2).$$

The red terms are cusp forms, and the orange terms converges to 1. Hence the limit is $\frac{36/\pi^2}{2} = \frac{18}{\pi^2}$. □

Proof of the monotonicity.

It is enough to show that

$\mathcal{L}_{1,0} = F'G - FG' = (\partial_{10}F)G - F(\partial_{10}G)$ is positive. We have

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$$\partial_{10}^2 F - \frac{\mathbf{5}}{\mathbf{6}} E_4 F = 172800 \Delta X_{4,2} > 0,$$
$$\partial_{10}^2 G - \frac{\mathbf{5}}{\mathbf{6}} E_4 G = -640 \Delta H_2 < 0$$

which gives

$$\partial_{22}\mathcal{L}_{1,0} = (\partial_{10}^2 F)G - F(\partial_{10}^2 G) = \Delta(172800 X_{4,2} G + 640 H_2 F) > 0$$

and so $\mathcal{L}_{1,0} > 0$. □

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Easy to be checked by Sage (or less easily by hands).

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$$\partial_{14}^2 F - \frac{\mathbf{14}}{\mathbf{9}} E_4 F = c \Delta X_{8,2},$$
$$\partial_{14}^2 G - \frac{\mathbf{14}}{\mathbf{9}} E_4 G = 0$$

for $c = 548674560$.

$d = 24$: harder inequality

The last inequality

$$t^{10}\left(-\frac{F(i/t)}{\Delta(i/t)^2}+\frac{432}{\pi^2}\frac{G(i/t)}{\Delta(i/t)^2}\right)\geq\frac{725760}{\pi}e^{2\pi t}\left(t-\frac{10}{3\pi}\right)$$

is more complicated, but we can prove it as follows:

[2] when $t \geq \frac{10}{3\pi}$. $0 < t \leq \frac{10}{3\pi}$ follows from the "hard" inequality.

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- The reduced inequality follows from monotonicity of it again, which is equivalent to (for $0 < t < \frac{3\pi}{10}$)

$$\mathcal{L}_{1,0}(it) - 725760\Delta(it)\left[(\partial_{12}G)(it)\left(\tfrac{1}{\pi t^3}-\tfrac{10}{3\pi^2 t^2}\right)-G(it)\left(\tfrac{3}{2\pi^2 t^4}-\tfrac{10}{3\pi^3 t^3}\right)\right] > 0$$

[2] when $t \geq \frac{10}{3\pi}$. $0 < t \leq \frac{10}{3\pi}$ follows from the “hard” inequality.

$d = 24$: harder inequality

- Use Serre derivative trick again; Serre derivative of the above function factors nicely and it reduces to the positivity of $7560 X_{8,2}(it) - \frac{37 E_4(it) - E_2(it)^2}{24}\left(\frac{1}{\pi t^3} - \frac{10}{3\pi^2 t^2}\right) - E_2(it)\left(\frac{3}{4\pi^2 t^4} - \frac{5}{3\pi^3 t^3}\right) + \left(\frac{3}{\pi^3 t^5} - \frac{5}{\pi^4 t^4}\right).$

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- If we denote this as $h(t)$, then $t^{-8} h(1/t)$ can be written as

$$7560 X_{8,2}(it) + \frac{1}{\pi t} \left[\left(\frac{3}{10} - \frac{1}{\pi t} \right) J_1(it) + \frac{3}{40} J_2(it) + \frac{7}{4} J_3(it) \right]$$

with

$$J_1 = \frac{5}{36} E_2^2 + \frac{1}{9} E_4 - \frac{1}{4} E_2, \quad J_2 = E_2 - E_6, \quad J_3 = E_2 E_4 - \frac{1}{10} E_6 - \frac{9}{10} E_4$$

which are all positive (for $t \geq \frac{10}{3\pi}$).

Future works

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- Make a formalization of the proof (e.g. in Lean) easier?

Future works

- What *are* the (completely) positive forms?
- Possible applications in other LP problems?
  - Dual LP, uncertainty principle, etc.
  - Any results that are "uniform" in dimensions?
- Make a formalization of the proof (e.g. in Lean) easier?
  - thefundamentaltheor3m / **Sphere-Packing-Lean** (WIP)

Paper: arxiv.org/abs/2406.14659

Code: github.com/seewoo5/posqmf

Thank you!

...Do I have more time?

Some possible explanations of mysterious numbers in identities

For $d = 24$, why we take $k = 14$ to prove $F(it) > 0$ using $\partial_{14}F(it) > 0$?

- In general, we have the following theorem: if F has weight w and depth s, then $\partial_{w-s}F$ has weight $w+2$ and **depth** $\leq s$. Our F has weight 16 and depth 2, so $k = 16 - 2$ is something special that we can try to see.

Some possible explanations of mysterious numbers in identities

Where the 5/6 of

$$\partial_{10}^2 F - \mathbf{\frac{5}{6}} E_4 F = 172800 \Delta X_{4,2} > 0,$$
$$\partial_{10}^2 G - \mathbf{\frac{5}{6}} E_4 G = -640 \Delta H_2 < 0$$

comes from?

- The differential operator $\partial_k^2 - \frac{k(k+2)}{144} E_4$ first appears in the paper "Supersingular j-invariants, hypergeometric series, and Atkin's orthogonal polynomials" by Kaneko–Zagier. Any possible conceptual connections/explanations?

Proof of Kaneko–Koike conjecture for $s = 1$

The new recurrence relations

$$X'_{w,1} = \frac{5w}{72} X_{6,1} X_{w-4,1} + \frac{7w}{72} X_{8,1} X_{w-6,1}$$
$$X'_{w+2,1} = \frac{5w}{72} X_{6,1} X_{w-2,1} + \frac{7w}{12} X_{8,1} X_{w-4,1}$$
$$X'_{w+4,1} = 240 X_{6,1} X_{w,1} + \frac{7w}{72} X_{8,1} X_{w-2,1} + \frac{5w}{72} X_{10,1} X_{w-4,1}$$

are conjectured based on Sage computations: If you try to express LHS with linear combination of the terms in RHS, then you get very simple positive rational coefficients, where the general formula can be easily guessed.

More comments on future works

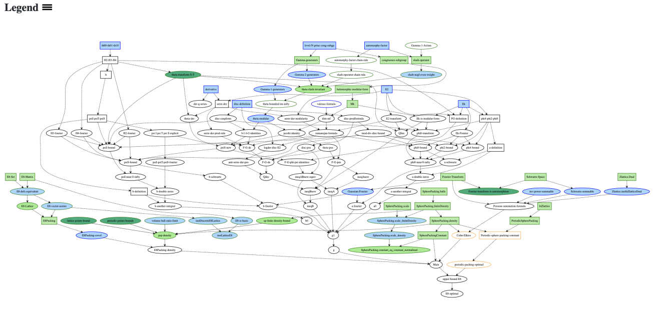
- What *are* the (completely) positive forms?
  - Counting functions? (Kaneko–Zagier) d-th coefficient of $X_{6,3}$ counts the number of simply ramified coverings of genus 2 and degree d of an elliptic curve over $\mathbb{C}$.
  - Geometric meaning? (Movasati) Quasimodular forms can be interpreted as sections of *jet bundles* on modular curves.
  - What are the "generators" of $\mathcal{QM}_{w,s}^{+}$ and $\mathcal{QM}_{w,s}^{++}$?
  - *Positivstellensatz* of quasimodular forms?
    - **Conjecture** (L., I don't believe) F is completely positive if and only if $F^{(r)}$ can be expressed in terms of extremal forms for some r.

More comments on future works

- Possible applications in other LP problems?
    - (Cohn–Triantafillou) Dual LP
    - (Brougain–Clozel–Kahane, Cohn–Gonçalves) Uncertainty principle
    - Any results that are "uniform" in dimensions?
        - Feigenbaum–Grabner–Hardin, *Eigenfunctions of the Fourier Transform with specified zeros*
        - **Theorem** (L.) Kaneko–Koike conjecture for $s = 2$ implies positivity of FGH's "$(-1)^{d/4}$" family of modular forms.

More comments on future works

- Make a formalization of the proof (e.g. in Lean) easier?
  - Leading by Sidharth Hariharan (for their master's thesis!) and including Chris Birkbeck, Gareth Ma, Maryna Viazovska, ...
  - Blueprint:



  - 1 year for completion?