Positive Quasimodular Forms and Linear Programming Bounds

by

Seewoo Lee

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Abstract

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Viazovska resolved the 8-dimensional sphere packing problem by constructing the *magic function* for the Cohn–Elkies linear programming bound, proving that the E_8 lattice packing is the densest possible packing. Soon after, Cohn, Kumar, Miller, Radchenko, and Viazovska gave a similar proof for the 24-dimensional case, showing that the Leech lattice gives the densest possible packing. One of the main steps of the proof is to verify nonpositivity and nonnegativity of the functions and their Fourier transforms, which reduces to inequalities for certain quasimodular forms. The original proofs by Viazovska and by Cohn et al. are based on interval arithmetic and Sturm's bound, which are numerical in nature. Later, Romik gave an alternative proof of the inequalities in dimension 8.

In this thesis, we give new *algebraic* proofs of the inequalities for both dimensions 8 and 24. In particular, we develop a theory of *positive* and *completely positive quasimodular forms*, and study how positivity interacts with derivatives and Serre derivatives of quasimodular forms. This theory is simple but powerful enough to give short proofs of the quasimodular form inequalities. We also find that the corresponding quasimodular forms are closely related to the *extremal quasimodular forms* by Kaneko and Koike. Along the way, we also prove that the depth 1 extremal quasimodular forms are completely positive, i.e. all the Fourier coefficients are positive, which resolves Kaneko and Koike's conjecture in this case.

The application of the theory of positive quasimodular forms is not limited to the sphere packing problem. We also study the monotonicity of functions of the form $t^m F(it)$ for a quasimodular form F and $t > 0$, which gives a new proof of one of the inequalities in Cohn et al.'s work on the universal optimality of E_8 and the Leech lattice, and also provides a way to construct positive quasimodular forms of higher levels. We also study

higher-level analogues of extremal quasimodular forms by Sakai and Tsutsumi, focusing on the positivity and integrality of their Fourier coefficients in the case of depth 1 and level $\Gamma_0(N)$ when $N = 2, 3, 4$, and also depth 2 and level $\Gamma_0(2)$.

Finally, we give new lower and upper bounds for Bourgain, Clozel, and Kahane's sign uncertainty principle in certain dimensions that are multiples of 4. For the upper bounds, we prove that the Fourier eigenfunctions constructed by Feigenbaum, Grabner, and Hardin are nonnegative, which gives improved upper bounds for $\mathrm{A}_+(d)$ for dimensions $d \leq 36000$. For the lower bounds, we follow Cohn and Gonçalves' approach and use summation formulas for radial Schwartz functions associated with extremal Eisenstein series, giving improved lower bounds for $\mathrm{A}_{(-1)^{d/4+1}}(d)$ for $d \leq 10000$.

To my parents and grandparents

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Chapter 1

Introduction

In this chapter, we give a brief overview of the history of the sphere packing problem and the main contributions of this thesis.

1.1 The story of the project

Before moving on to the main content, let me describe how this project began.[1]

In 2016, when I was a senior undergraduate student at POSTECH, I heard the news that the 8-dimensional sphere packing problem had been solved by Maryna Viazovska [80]. There was a weekly undergraduate seminar run by a mathematics club called MARCUS (MAthematics Research Club for Undergraduate Students), and I gave a talk on the paper because I had just started to become interested in modular forms and this seemed to be a striking application of them. I was not able to understand most of the paper at that time, but I did notice that one key step involved inequalities for modular forms proved by numerical methods, more precisely using interval arithmetic. Later, I heard that Viazovska won a Fields Medal for this work in 2022.

On September 11, 2023, Dan Romik came to Berkeley to give a talk at the Berkeley RTG seminar on his new proof of the modular form inequalities for the 8-dimensional sphere packing problem [65]. After a preparatory talk on the background of sphere packing, he presented the proof and ended by posing, as an open problem, finding a human proof of the corresponding inequalities in dimension 24. I asked him afterward whether he had thought about it, and he said no, but that it would be a nice project for graduate

[1] This section is motivated by the similarly titled section in the introduction of Aaron Landesman's thesis.

students. The problem seemed intriguing, so I began working on it as a side project. After a few weeks, it had displaced my main project, but I had also proved the first of the dimension-24 inequalities. Progress then stalled for a few months, so I decided to return to the dimension 8 case, even though two proofs were already known. Luckily, I found a third proof of the dimension-8 inequalities. The argument generalized to dimension 24, and I was elated for a few hours thinking I had solved the problem, until I realized that one inequality still remained—and that it was considerably harder than the others. It took a few more months to find a satisfactory proof of the third inequality, and by then it was clear that the project had become central to my thesis.

After I uploaded the paper to arXiv [49], Yaël Dillies found it and shared it on the Xena Discord server, a community for students in London who are interested in Lean. By coincidence, Sidharth Hariharan, who was a master's student at Imperial College London, was working on a formalization of Viazovska's proof of the 8-dimensional sphere packing problem, and he was also a member of the Discord server. He eventually sent me a message, and I joined the project, along with Chris Birkbeck, Gareth Ma, and Bhavik Mehta. After working on the project for a year, we made it public at the Big Proof conference on June 13, 2025. The project is still ongoing, and many people are contributing to it; see Appendix B for details.

1.2 High-dimensional sphere packings and quasimodular forms

This section reviews the sphere packing problem, focusing on Viazovska's and Cohn et al.'s resolutions of the 8- and 24-dimensional cases [80, 19]. A detailed exposition can be found in Chapter 2.

The sphere packing problem in dimension d asks the following: *what is the densest way of packing Euclidean space $\mathbb{R}^d$ with non-overlapping unit balls, and what is the maximum density Δ_d?* The case $d = 1$ is trivial, since intervals of the same length can tile the real line. For $d = 2$, the hexagonal lattice packing is optimal with density $\Delta_2 = \frac{\pi}{\sqrt{12}}$; this result is often attributed to Thue [76], although Tóth [27] is considered to have given the first rigorous proof. In dimension $d = 3$, Kepler conjectured in 1611 that the densest possible packing can be obtained by stacking hexagonal packings with density $\Delta_3 = \frac{\pi}{\sqrt{18}}$. This remained a conjecture for almost 400 years, until Thomas Hales gave a rigorous proof in 1998. The paper was eventually accepted by the Annals of Mathematics in 2005 [35], but some uncertainty remained because the proof was heavily computer-assisted and was based on 100000 linear programming problems. Eventually, Hales decided to *formalize* the proof on computers through the *Flyspeck* project, an expansion of the acronym FPK standing for *Formal Proof of Kepler*. It took about 10 years to complete the formalization of the proof in Isabelle and HOL Light [34].

For higher dimensions, it is known that D_4, D_5, E_6, E_7, E_8, and the Leech lattice Λ_{24} are optimal among *lattice packings* in dimensions 4, 5, 6, 7, 8, and 24, respectively [47, 46, 5, 79, 15]. However, optimality among *all* packings in higher dimensions remains an open problem. One effective way to prove an upper bound is Cohn and Elkies' *linear programming bound* [13], which roughly says that a suitable function on $\mathbb{R}^d$ satisfying certain linear constraints yields an upper bound (see Theorem 2.4.1 for the precise statement). Their numerical experiments suggested that there may exist certain *magic functions* in dimensions 8 and 24 that give upper bounds matching the lower bounds given by the densities of the E_8 and Λ_{24} packings. Number theorists had hoped to construct these optimal functions using *modular forms* [11], one of the central classes of functions in number theory, but no explicit construction was known until 2016. Note that the Cohn–Elkies bound is known to be suboptimal for dimensions $d = 3, 4, 5, 6, 7$, i.e. there does *not* exist a magic function in these dimensions, as proved using *dual* linear programming bounds [17, 51].

On π-Day, March 14, 2016, Maryna Viazovska uploaded a 23-page paper to arXiv constructing the optimal function in dimension 8 as an integral transform of certain quasimodular forms [80]. Peter Sarnak was quoted as follows in an interview with Quanta Magazine [73]: "It's stunningly simple, as all great things are. You just start reading the paper and you know this is correct." After the paper appeared, she teamed up with Henry Cohn, Abhinav Kumar, Stephen D. Miller, and Danylo Radchenko to construct the magic function in dimension 24 in a week, proving that the Leech lattice packing is optimal [19]. More stories about the collaboration can be found in the interview by de Laat and Vallentin [48].

1.3 Positive quasimodular forms and a new proof of the inequalities

The final step in these sphere packing proofs is to verify certain sign inequalities for the magic functions. Viazovska and Cohn et al. reduced this verification to inequalities for quasimodular forms. In dimension 8, Viazovska proved them using interval arithmetic; in dimension 24, Cohn et al. used a different numerical method based on Sturm's bound for polynomials. Given the conceptual nature of the magic functions themselves, it is natural to ask whether these inequalities also admit a more conceptual proof. In [48], Miller said that "Though I think there will eventually be a slick proof that can be written by hand, computers were completely essential in this story."

Romik gave such a proof for dimension 8 [65], which he called "calculator-assisted" rather than "computer-assisted" because the proof only requires comparing values of certain quasimodular forms at $z = i$. A new proof of the 24-dimensional case remained open at the time, and it was the main motivation for this work.

In this thesis, we first develop a theory of positive and completely positive quasimodular forms (Chapter 3). These are quasimodular forms that are positive on the imaginary axis or have nonnegative Fourier coefficients, respectively. We study how these properties interact with derivatives and Serre derivatives (Propositions 3.2.1, 3.2.2, 3.2.4, and 3.3.4). Before this work, positivity of quasimodular forms had received little direct attention, even though sign changes of Fourier coefficients of modular forms are widely studied (see, e.g., [45]). The basic results are simple, but they are strong enough to give a new, *algebraic* proof of the quasimodular form inequalities in both dimensions 8 and 24 (Chapter 5). We also find that the relevant quasimodular forms are closely related to Kaneko and Koike's *extremal quasimodular forms* [40] through several modular linear differential equations ((5.24) and (5.36)). As a byproduct, we resolve a conjecture of Kaneko and Koike on the complete positivity of the extremal quasimodular forms of depth 1 (Corollary 4.2.2). This part of the work is also available as [49] on arXiv.

1.4 More applications

The theory of positive quasimodular forms also has several applications beyond the original sphere packing inequalities.

An inequality for the universal optimality of the Leech lattice

First, Cohn, Kumar, Miller, Radchenko, and Viazovska proved that the E_8 and Leech lattice packings are *universally optimal* [20], meaning that they minimize the energy for a large class of potential functions. Their proof combines the linear programming bound for potential energy by Cohn and Kumar [16] with Fourier interpolation formulas for radial Schwartz functions on $\mathbb{R}^8$ and $\mathbb{R}^{24}$. More precisely, they proved that there are interpolation basis functions $\{a_n, b_n, \tilde{a}_n, \tilde{b}_n\}_{n\geq n_0}$ such that every radial Schwartz function f can be expressed as

$$f(\mathbf{x}) = \sum_{n\geq n_0} f(\sqrt{2n})a_n(\mathbf{x}) + \sum_{n\geq n_0} f'(\sqrt{2n})b_n(\mathbf{x}) \\ + \sum_{n\geq n_0} \widehat{f}(\sqrt{2n})\tilde{a}_n(\mathbf{x}) + \sum_{n\geq n_0} \widehat{f}'(\sqrt{2n})\tilde{b}_n(\mathbf{x}),$$

where $n_0 = 1$ for $d = 8$ and $n_0 = 2$ for $d = 24$, and the series converge absolutely. The construction starts from generating functions $F, \widetilde{F} : \mathbb{H} \times \mathbb{R}^d$,

$$F(\tau, \mathbf{x}) = \sum_{n\geq n_0} a_n(\mathbf{x})e^{2\pi i n\tau} + 2\pi i\tau \sum_{n\geq n_0} \sqrt{2n}b_n(\mathbf{x})e^{2\pi i n\tau}$$
$$\widetilde{F}(\tau, \mathbf{x}) = \sum_{n\geq n_0} \tilde{a}_n(\mathbf{x})e^{2\pi i n\tau} + 2\pi i\tau \sum_{n\geq n_0} \sqrt{2n}\tilde{b}_n(\mathbf{x})e^{2\pi i n\tau},$$

for which the interpolation formula for the complex Gaussian $\mathbf{x} \mapsto e^{\pi i\tau\|\mathbf{x}\|^2}$ reduces to the identity

$$F(\tau, \mathbf{x}) + \left(\frac{i}{\tau}\right)^{\frac{d}{2}} \widetilde{F}\left(-\frac{1}{\tau}, \mathbf{x}\right) = e^{\pi i\tau\|\mathbf{x}\|^2}.$$

As with the magic functions for sphere packing, F and $\widetilde{F}$ are constructed as integral transforms of two-variable kernel functions $\mathcal{K}(\tau, z)$ and $\widehat{\mathcal{K}}(\tau, z)$ on $\mathbb{H} \times \mathbb{H}$, which are expressed in terms of quasimodular forms and logarithms of the modular lambda function. Universal optimality then follows from positivity of the kernel functions on the imaginary axis. Verifying this positivity reduces to complicated inequalities of quasimodular forms and polynomials, and the original proof is heavily computer-assisted.

In Chapter 6, we give a new proof of inequality (3b) in [20, p. 1067] which does not require any numerical methods (Lemma 6.6.1). The key point is to study monotonicity of the functions of the form

$$t \mapsto t^m F(it)$$

when $m \geq 0$ and F is a quasimodular form. We give several sufficient conditions for monotonicity (Propositions 6.2.1 and 6.2.10), and show that inequality (3b) is equivalent to such a monotonicity statement with $m = 11$ and $F = X_{12,1}$, the extremal quasimodular form of weight 12 and depth 1 (Lemma 6.6.1). The required monotonicity follows from Corollary 6.2.7. The same idea also constructs a new family of positive quasimodular forms of higher levels (Example 6.4.3) and gives another proof of the second quasimodular form inequality for the sphere packing problem in dimension 24 (Section 6.5 and Appendix D). This part of the work is also available as [50] on arXiv.

Higher level extremal quasimodular forms

In Chapter 4, we study positivity of level 1 extremal quasimodular forms of Kaneko–Koike [40]. In Chapter 7, we study analogues of their results for extremal quasimodular forms of level $\Gamma_0(N)$ for $N = 2, 3, 4$, introduced by Sakai and Tsutsumi [67]. We first prove uniqueness of extremal quasimodular forms of level $\Gamma_0(N)$ and depth s for $(N, s) \in \{(2,1),(2,2),(3,1),(4,1)\}$ by adapting Pellarin's argument [61] (Corollary 7.2.4).

For each pair (N, s) listed above, we derive new recurrence relations (Theorems 7.3.7 and 7.4.5, and Proposition 7.3.16). We use them to prove positivity of the depth 1 forms $\mathcal{D}_w$ and $\mathcal{F}_w$ (level $\Gamma_0(2)$ and $\Gamma_0(4)$, respectively; see Propositions 7.3.8 and 7.5.3). In all cases, we also classify those with integral Fourier coefficients (Theorems 7.3.14, 7.4.7, and 7.3.19), extending the level 1 results of Kaminaka–Kato [39] and Nakaya [57]. For $(N, s) = (2, 2)$, we further classify the completely positive forms (Corollary 7.3.17), showing that only finitely many weights give completely positive forms, in contrast to the depth 1 case.

New bounds for the sign uncertainty principle

As a final application, we give new lower and upper bounds for the sign uncertainty principle of Bourgain, Clozel, and Kahane [7]. The sign uncertainty principle concerns the simultaneous eventual signs of a function and its Fourier transform. More precisely, for a real-valued function f on $\mathbb{R}^d$ satisfying natural sign conditions at the origin, Bourgain, Clozel, and Kahane showed that the product $r(f)r(\widehat{f})$ is bounded below by a positive constant, where $r(f)$ and $r(\widehat{f})$ denote the radii beyond which f and $\widehat{f}$ are nonnegative. We denote the square root of the infimum of $r(f)r(\widehat{f})$ by $\mathrm{A}_+(d)$; there is also a negative version $\mathrm{A}_-(d)$. The only known exact value is $\mathrm{A}_+(12) = \sqrt{2}$, due to Cohn and Gonçalves [14].

In Chapter 9, we prove new upper and lower bounds in dimensions that are multiples of 4 (Theorems 9.0.1 and 9.0.2), improving earlier results of Bourgain–Clozel–Kahane [7], Gonçalves–Silva–Steinerberger [30], Cohn–Gonçalves [14], and Edwin [25]. For the upper bound, we show that for multiples of 4 with $d \le 36000$,

$$\mathrm{A}_+(d) \le \sqrt{2\left\lfloor \frac{d}{16} \right\rfloor + 2}.$$

The asymptotic constant $1/\sqrt{8} \approx 0.3535$ is strictly smaller than the constant $1/\sqrt{2\pi} \approx 0.3989$ in Bourgain–Clozel–Kahane's bound. This upper bound follows from the positivity of the Feigenbaum–Grabner–Hardin Fourier eigenfunctions proved in Chapter 8 (Theorems 8.1.5 and 8.2.8), together with a computer check for the nonpositivity of their values at the origin.

For the lower bound, Theorem 9.0.2 shows that for multiples of 4 with $d \leq 10000$,

$$A_{(-1)^{d/4+1}}(d) \geq \begin{cases} \sqrt{2\left\lfloor \frac{d}{24} \right\rfloor + 2} & d \not\equiv 4 \pmod{24}, \\ \sqrt{2\left\lfloor \frac{d}{24} \right\rfloor} & d \equiv 4 \pmod{24}. \end{cases}$$

The asymptotic constant $1/\sqrt{12} \approx 0.2887$ improves Edwin's bound of $1/\sqrt{4\pi} \approx 0.2821$ in these dimensions. The proof adapts the summation-formula method of Cohn and Gonçalves, using radial Schwartz summation formulae associated with *extremal Eisenstein series*: the unique level-1 modular forms whose Fourier expansions begin with 1 followed by as many zeros as possible. For small weights, the required sign pattern of their Fourier coefficients is established by explicit identities involving the extremal quasimodular forms from Chapter 4 (Proposition 9.4.4); for larger weights, it follows from explicit coefficient bounds of Jenkins and Rouse [38] (Theorem 9.4.3) together with a finite computer verification.

1.5 Sage and Lean

Most of the results in this thesis are inspired by experiments with Sage [75]. For example, the new proof of the quasimodular form inequalities was inspired by Figure 5.1, which was plotted using Sage, and various recurrence relations for extremal quasimodular forms such as those in Theorem 4.2.1 were also conjectured from Sage experiments. Details on the Sage code can be found in Appendix A.

In addition, some of the results in this thesis are formalized in Lean 4 [55]. Most notably, the quasimodular form inequalities for the sphere packing problem in dimension 8 are formalized as part of the `Sphere-Packing-Lean` project mentioned above; details can be found in Appendix B. Some additional results on positivity of certain quasimodular forms and inequalities such as (6.2) are also formalized; these are all elementary but tedious to verify by hand.

All Sage and Lean code is available in the following GitHub repository:

https://github.com/seewoo5/posqmf

1.6 Disclosure of AI usage

At the time of writing this thesis, large language models (LLMs) such as `ChatGPT` (OpenAI), `Claude` (Anthropic), and `Gemini` (Google) have become powerful tools for various tasks, including editing text, locating literature, writing code, and even providing mathematical proofs. For transparency, we describe the use of AI in this thesis.

Most of the core mathematical work was done by the author, with LLMs used primarily to fix typos and grammatical errors. However, there are a few instances where LLMs were used to assist in proving lemmas. For example, a key step in the proof of Lemma 6.1.5, showing positivity of a certain elementary but complicated function, was suggested by `ChatGPT-5.2 Pro`. LLMs were also used to compute certain decompositions of quasimodular forms to establish positivity or negativity of their Fourier coefficients, as in Propositions 4.5.1 and 7.3.20 and Corollary 7.3.17, and to identify errors in the original proofs of these results. In many of these cases, the author provided a proof sketch, and LLMs were used to fill in the details and write them up in a more polished form.

In some cases, proofs suggested by LLMs were incorrect and had to be corrected by the author. For example, the original proofs of Lemma 6.1.5 suggested by `ChatGPT-5.2 Pro` and `Gemini Pro` were both incorrect. This motivated the author to formalize some of these proofs in Lean 4 to verify their correctness. Most of the Lean code was written by AIs, including `Claude Opus 4.7`, `Codex-5.3` (OpenAI), and `AxiomProver` (Axiom Math), with the formalized statements carefully checked by the author to ensure that they match the intended mathematical statements.

Chapter 2

Preliminaries

In this chapter, we review the basics of modular forms and quasimodular forms with examples. We also give a brief introduction to Viazovska's and Cohn et al.'s proofs of the 8- and 24-dimensional sphere packing problems [80, 19].

2.1 Modular forms

For any function $f : \mathbb{H} \to \mathbb{C}$ defined on the complex upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ and any integer w, we define the action of $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ as

$$(f|_w \gamma)(z) := (cz+d)^{-w} f\left(\frac{az+b}{cz+d}\right), \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

We denote by $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ the matrices that generate $\mathrm{SL}_2(\mathbb{Z})$.

Eisenstein series are among the most famous examples of modular forms, and they are also easy to construct. For an even integer $w \geq 4$, consider the sum

$$\sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} (1|_w \gamma)(z) = \sum_{\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})} \frac{1}{(cz+d)^w}$$

which would be invariant under the slash action, *if it converged*. However, this sum fails to converge. In fact, the subgroup $\Gamma_\infty = \{\left[\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right] : n \in \mathbb{Z}\}$ is the stabilizer of the constant function, so we can take a quotient and the sum

$$E_w(z) := \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} (1|_w \gamma)(z) = \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{1}{(cz+d)^w} \tag{2.1}$$

indeed converges absolutely, provided $w > 2$. This defines the Eisenstein series of weight w. The Fourier expansion of E_w is given by [9, p. 16]

$$E_w(z) = 1 - \frac{2w}{B_w} \sum_{n\geq 1} \sigma_{w-1}(n) q^n, \tag{2.2}$$

where B_w is the w-th Bernoulli number, defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{k\geq 0} B_k \frac{x^k}{k!} \tag{2.3}$$

and $\sigma_a(n) := \sum_{d|n} d^a$ is the divisor power sum function. For example, the weight 4 and weight 6 Eisenstein series are

$$E_4(z) = 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n \tag{2.4}$$

$$E_6(z) = 1 - 504 \sum_{n\geq 1} \sigma_5(n) q^n \tag{2.5}$$

satisfying

$$E_4(z+1) = E_4(z), \tag{2.6}$$

$$E_6(z+1) = E_6(z), \tag{2.7}$$

$$z^{-4} E_4\left(-\frac{1}{z}\right) = E_4(z), \tag{2.8}$$

$$z^{-6} E_6\left(-\frac{1}{z}\right) = E_6(z). \tag{2.9}$$

The ring of even-weight level-1 modular forms is isomorphic to the polynomial ring $\mathbb{C}[E_4, E_6]$.

A modular form is called a *cusp form* if it vanishes at all cusps, i.e. its Fourier expansions have no constant terms. For example, every Eisenstein series E_w has a nonzero constant term 1 (2.2), hence it is not a cusp form. The first nonzero cusp form is the *discriminant form*

$$\Delta(z) = \frac{E_4^3 - E_6^2}{1728} = q \prod_{n\geq 1} (1 - q^n)^{24} = \eta(z)^{24} \tag{2.10}$$

which is a cusp form of weight 12 and level 1. Here $\eta(z) = q^{1/24} \prod_{n\geq 1}(1 - q^n)$ is the *Dedekind eta function*, with $q^{1/24} = e^{2\pi i z/24}$.

Another interesting class of examples arises from lattices [21]. Let $\Lambda \subset \mathbb{R}^d$ be a d-dimensional lattice, i.e. a free $\mathbb{Z}$-submodule of rank d. One can associate to Λ a *theta function* that counts the norms of the vectors in Λ, namely

$$\Theta_\Lambda(z) = \sum_{v\in\Lambda} q^{\frac{\langle v,v\rangle}{2}}. \tag{2.11}$$

Let $\Lambda^* := \{w \in \mathbb{R}^d : \langle v, w\rangle \in \mathbb{Z}, \forall v \in \Lambda\}$ be the *dual lattice* of Λ. One has a Poisson summation formula: for any *admissible function* $f : \mathbb{R}^d \to \mathbb{R}$ (a slightly weaker condition than being Schwartz; see [13, Definition 2.2] for details), we have

$$\sum_{v\in\Lambda} f(v) = \frac{1}{|\Lambda|}\sum_{w\in\Lambda^*} \widehat{f}(w) \tag{2.12}$$

where $\widehat{f}$ is the Fourier transform

$$\widehat{f}(\mathbf{y}) := \int_{\mathbb{R}^d} f(\mathbf{x})e^{-2\pi i\langle \mathbf{x},\mathbf{y}\rangle}\mathrm{d}\mathbf{x} \tag{2.13}$$

and $|\Lambda|$ is the volume of the fundamental domain $\mathbb{R}^d/\Lambda$. When $\Lambda = \mathbb{Z}$, this recovers the usual Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) \tag{2.14}$$

for *nice* functions $f : \mathbb{R} \to \mathbb{R}$. By taking $f(\mathbf{x}) := e^{\pi i z|\mathbf{x}|^2}$ for $\mathbf{x} \in \mathbb{R}^d$, we get

$$\Theta_{\Lambda^*}(z) = |\Lambda|^{\frac{1}{2}}\left(\frac{i}{z}\right)^{\frac{d}{2}}\Theta_\Lambda\left(-\frac{1}{z}\right). \tag{2.15}$$

In particular, when Λ is self-dual (i.e. $\Lambda^* = \Lambda$), we get

$$\Theta_{\Lambda}(z) = |\Lambda|^{\frac{1}{2}}\left(\frac{i}{z}\right)^{\frac{d}{2}}\Theta_\Lambda\left(-\frac{1}{z}\right), \tag{2.16}$$

which implies that Θ_Λ is a modular form of weight $d/2$ (in the sense of Shimura [70]). Its Fourier expansion has the form

$$\Theta_\Lambda(z) = \sum_{n\geq 0} r_\Lambda(n) q^{\frac{n}{2}} \tag{2.17}$$

where $r_\Lambda(n) := \#\{v \in \Lambda : \langle v, v\rangle = n\}$ is the number of vectors in Λ with (squared) norm n. If Λ is even and unimodular, i.e. $\langle v, v\rangle \in 2\mathbb{Z}$ for all $v \in \Lambda$ and $|\Lambda| = 1$, then all the exponents in (2.17) are integers, and Θ_Λ is a modular form of weight $d/2$ for $\mathrm{SL}_2(\mathbb{Z})$. For $d = 1$ and $\Lambda = \mathbb{Z}$, this gives one of the *Jacobi theta functions*

$$\Theta_3(z) := \Theta_{\mathbb{Z}}(z) = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}}, \tag{2.18}$$

which satisfies

$$\Theta_3\left(-\frac{1}{z}\right) = \left(\frac{z}{i}\right)^{\frac{1}{2}} \Theta_3(z). \tag{2.19}$$

The other two theta functions Θ_2 and Θ_4 are defined as

$$\Theta_2(z) := \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \tag{2.20}$$

$$\Theta_4(z) := \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n^2}{2}} \tag{2.21}$$

which transform as

$$\Theta_2(z+1) = \sqrt{i}\Theta_2(z) \tag{2.22}$$

$$\Theta_3(z+1) = \Theta_4(z) \Leftrightarrow \Theta_4(z+1) = \Theta_3(z) \tag{2.23}$$

$$\Theta_2\left(-\frac{1}{z}\right) = \left(\frac{z}{i}\right)^{\frac{1}{2}} \Theta_4(z) \Leftrightarrow \Theta_4\left(-\frac{1}{z}\right) = \left(\frac{z}{i}\right)^{\frac{1}{2}} \Theta_2(z). \tag{2.24}$$

Jacobi's famous identity relates these three functions:

$$\Theta_3^4 = \Theta_2^4 + \Theta_4^4 \tag{2.25}$$

For simplicity, we will denote the fourth powers of Jacobi's theta functions by

$$H_2 := \Theta_2^4, \quad H_3 := \Theta_3^4, \quad H_4 := \Theta_4^4. \tag{2.26}$$

Their Fourier expansions are given by

$$H_2(z) = 2\sum_{n\geq 0} r_4(2n+1) q^{n+\frac{1}{2}}, \tag{2.27}$$

$$H_3(z) = 1 + \sum_{n\geq 1} r_4(n) q^{\frac{n}{2}}, \tag{2.28}$$

$$H_4(z) = 1 + \sum_{n\geq 1} (-1)^n r_4(n) q^{\frac{n}{2}}, \tag{2.29}$$

where $r_4(k) := \#\{\mathbf{x} \in \mathbb{Z}^4 : \|\mathbf{x}\|^2 = k\}$. These are modular forms of weight 2 and level $\Gamma(2)$, satisfying the following transformation laws:

$$H_2|T = -H_2, \quad H_3|T = H_4, \quad H_4|T = H_3, \tag{2.30}$$

$$H_2|S = -H_4, \quad H_3|S = -H_3, \quad H_4|S = -H_2. \tag{2.31}$$

Theta functions and E_4, E_6, Δ are related as follows:

$$E_4 = \frac{1}{2}(H_2^2 + H_3^2 + H_4^2) = H_2^2 + H_2H_4 + H_4^2 \tag{2.32}$$

$$E_6 = \frac{1}{2}(H_2 + H_3)(H_3 + H_4)(H_4 - H_2) = \frac{1}{2}(H_2 + 2H_4)(2H_2 + H_4)(H_4 - H_2) \tag{2.33}$$

$$\Delta = \frac{1}{256}(H_2H_3H_4)^2. \tag{2.34}$$

Proofs of these identities that use only the dimension formula for level 1 modular forms are given in Appendix B.

A quotient of theta functions gives a modular lambda function of level $\Gamma(2)$, namely

$$\lambda = \frac{H_2}{H_3} = \frac{H_2}{H_2 + H_4} \tag{2.35}$$

which is known to be the Hauptmodul of the modular curve $X(2)$, i.e., a generator of the function field of $X(2)$ (see, e.g., [23]). Under the action of S, the lambda function transforms as

$$\lambda_S := \lambda|_0 S = \frac{H_4}{H_2 + H_4} = 1 - \lambda. \tag{2.36}$$

We denote its logarithm by $\mathcal{L}_S := \log \lambda_S$, which admits the Fourier expansion

$$\mathcal{L}_S(z) = -16 \sum_{k \geq 0} \frac{\sigma_1(2k+1)}{2k+1} q^{k+\frac{1}{2}} \tag{2.37}$$

and its derivative is

$$\mathcal{L}_S' = \frac{H_4'}{H_4} - \frac{H_2' + H_4'}{H_2 + H_4} = -\frac{1}{2}H_2. \tag{2.38}$$

This function will reappear in Chapter 8.

2.2 Quasimodular forms

A *quasimodular form* of weight w and depth $\le s$ for $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $F : \mathbb{H} \to \mathbb{C}$ that satisfies a transformation law of the form

$$(F|_w\gamma)(z) = \sum_{j=0}^{s} f_j(z) \left(\frac{c}{cz+d}\right)^j, \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma,$$

for some holomorphic functions $f_0, \dots, f_s$ on $\mathbb{H}$. In particular, a quasimodular form of depth 0 is an ordinary modular form.

The basic example is the weight 2 Eisenstein series, defined by the Fourier expansion

$$E_2(z) = 1 - 24 \sum_{n\ge 1} \sigma_1(n) q^n. \tag{2.39}$$

It fails to be a modular form because of the anomalous transformation

$$(E_2|_2S)(z) = z^{-2} E_2\left(-\frac{1}{z}\right) = E_2(z) - \frac{6i}{\pi z}, \tag{2.40}$$

so E_2 is a quasimodular form of weight 2 and depth 1. The (graded) ring of quasimodular forms for $\mathrm{SL}_2(\mathbb{Z})$ is isomorphic to $\mathbb{C}[E_2, E_4, E_6]$ [9, §5.1]. We define the *depth* of a quasimodular form as the highest degree of E_2 in its expression as a polynomial in E_2, E_4, and E_6. It is closed under differentiation

$$F' = DF := \frac{1}{2\pi i}\frac{\mathrm{d}F}{\mathrm{d}z} = q\frac{\mathrm{d}F}{\mathrm{d}q}, \quad \sum_n a_n q^n \mapsto \sum_n n a_n q^n \tag{2.41}$$

which increases depth by 1 and weight by 2. For the Eisenstein series, we have Ramanujan's identities [9, Proposition 15, p. 49]

$$E_2' = \frac{E_2^2 - E_4}{12}, \tag{2.42}$$

$$E_4' = \frac{E_2E_4 - E_6}{3}, \tag{2.43}$$

$$E_6' = \frac{E_2E_6 - E_4^2}{2}. \tag{2.44}$$

We write $\mathcal{QM}_w^s = \mathcal{QM}_w^s(\mathrm{SL}_2(\mathbb{Z}))$ for the space of quasimodular forms of weight w and depth $\le s$, and $\mathcal{M}_w := \mathcal{QM}_w^0$ for the space of genuine modular forms of weight w. We also recall the discriminant form $\Delta = (E_4^3 - E_6^2)/1728$ from (2.10), the unique normalized cusp form of weight 12 and level $\mathrm{SL}_2(\mathbb{Z})$.

For an integer k and a quasimodular form F, the weight k Serre derivative $\partial_k F$ of F is given by

$$\partial_k F := F' - \frac{k}{12} E_2 F.$$

For $F \in \mathcal{QM}_w^s$, $\partial_k F$ is *a priori* a quasimodular form of weight $w+2$ and depth $s+1$, but when $k = w - s$, Kaneko and Koike proved that it preserves the depth of quasimodular forms, i.e. $\partial_k F \in \mathcal{QM}_{w+2}^s$ [40, Proposition 3.3]. The Serre derivative is equivariant under the $\mathrm{SL}_2(\mathbb{Z})$-action in the sense that

$$\partial_k (F|_k \gamma) = (\partial_k F)|_{k+2}\gamma, \quad \forall \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

The Serre derivatives of Eisenstein series and Jacobi theta functions are given by:

$$\partial_1 E_2 = -\frac{1}{12} E_4, \quad \partial_4 E_4 = -\frac{1}{3} E_6, \quad \partial_6 E_6 = -\frac{1}{2} E_4^2 \tag{2.45}$$

and

$$\partial_2(H_2) = \frac{1}{6}(H_2^2 + 2H_2H_4), \tag{2.46}$$

$$\partial_2(H_3) = \frac{1}{6}(H_2^2 - H_4^2), \tag{2.47}$$

$$\partial_2(H_4) = -\frac{1}{6}(2H_2H_4 + H_4^2), \tag{2.48}$$

or equivalently,

$$H_2' = \frac{1}{6}(H_2^2 + 2H_2H_4 + E_2H_2) \tag{2.49}$$

$$H_3' = \frac{1}{6}(H_2^2 - H_4^2 + E_2(H_2 + H_4)) \tag{2.50}$$

$$H_4' = -\frac{1}{6}(2H_2H_4 + H_4^2 - E_2H_4) \tag{2.51}$$

More generally, the product rule gives

$$\partial_{2a+2b}(H_2^a H_4^b) = \frac{1}{6} H_2^a H_4^b((a - 2b)H_2 + (2a - b)H_4) \tag{2.52}$$

for nonnegative integers a, b.

The Serre derivative satisfies the product rule

$$\partial_{w_1+w_2}(FG) = (\partial_{w_1}F)G + F(\partial_{w_2}G). \tag{2.53}$$

In particular, we have

$$\partial_w(E_2F) = E_2(\partial_{w-1}F) - \frac{1}{12}E_4F, \tag{2.54}$$

$$\partial_w(E_4F) = E_4(\partial_{w-4}F) - \frac{1}{3}E_6F, \tag{2.55}$$

$$\partial_w(E_6F) = E_6(\partial_{w-6}F) - \frac{1}{2}E_4^2F, \tag{2.56}$$

which are useful for computations.

We also record the following identities for later use (Example 6.4.2 of Section 6.4):

Lemma 2.2.1.

$$6E_2(2z) = 4E_2(2z) + E_2\left(\frac{z}{2}\right) + E_2\left(\frac{z+1}{2}\right). \tag{2.57}$$

Proof. This is [66, Exercise 5.19]. For completeness, we include a proof. The main idea is to compare the Fourier coefficients of both sides. Comparing coefficients of $q^{n/2}$, (2.57) is equivalent to

$$6\sigma_1\left(\frac{n}{2}\right) = 4\sigma_1\left(\frac{n}{4}\right) + \sigma_1(n)(1+(-1)^n)$$

where $\sigma_1(a) = 0$ if $a \notin \mathbb{Z}_{\geq 0}$. When n is odd, $\sigma_1(n/2) = \sigma_1(n/4) = 0$ and $1 + (-1)^n = 0$, so the equality holds. When n is even, let $n = 2^k m$ where m is odd and $k \geq 1$. If $k = 1$, then $\sigma_1(n/2) = \sigma_1(m)$ and $\sigma_1(n/4) = 0$, and the equation reduces to $6\sigma_1(m) = 2\sigma_1(2m)$, which follows from the multiplicativity of σ_1 and $\sigma_1(2) = 3$. For $k \geq 2$, again using multiplicativity, we have

$$\sigma_1(n) = \sigma_1(2^k)\sigma_1(m) = (1 + 2 + 2^2 + \cdots + 2^k)\sigma_1(m) = (2^{k+1} - 1)\sigma_1(m)$$

and similarly for $\sigma_1(n/2)$ and $\sigma_1(n/4)$, and the equation reduces to $6(2^k - 1) = 4(2^{k-1} - 1) + 2(2^{k+1} - 1)$. □

2.3 Rankin–Cohen bracket

The Rankin–Cohen bracket is a family of bilinear operators that send a pair of modular forms to another modular form [64, 10]. Given modular forms f and g (possibly different weights) and nonnegative integers k, l, n, define $[f, g]_n^{(k,l)}$ as

$$[f, g]_n^{(k,l)} = \sum_{i+j=n} (-1)^i \binom{n+k-1}{j} \binom{n+l-1}{i} f^{(i)} g^{(j)} \tag{2.58}$$

where the derivatives are normalized as (2.41). The basic fact is that, when k and l are the weights of f and g respectively, $[f, g]_n^{(k,l)}$ is also a modular form of weight $k + l + 2n$. The first few are as follows:

$$\begin{aligned}
[f, g]_0^{(k,l)} &= fg \\
[f, g]_1^{(k,l)} &= kfg' - lf'g \\
[f, g]_2^{(k,l)} &= \binom{k+1}{2} fg'' - (k+1)(l+1)f'g' + \binom{l+1}{2} f''g \\
[f, g]_3^{(k,l)} &= \binom{k+2}{3} fg''' - \binom{k+2}{2}(l+2)f'g'' + (k+2)\binom{l+2}{2} f''g' - \binom{l+2}{3} f'''g.
\end{aligned}$$

There are several ways to prove the modularity of the Rankin–Cohen bracket, such as relating a generating series of $[-,-]_n$ to a Jacobi-like form [82], or writing these brackets as *canonical higher Serre derivatives* [56]. More generally, Martin and Royer proved that a slight modification of the Rankin–Cohen bracket acts on a pair of quasimodular forms without increasing their combined depth; if $f \in \mathcal{QM}_k^{\leq s}(\Gamma)$ and $g \in \mathcal{QM}_l^{\leq t}(\Gamma)$, then $[f, g]_n^{(k-s,l-t)} \in \mathcal{QM}_{k+l+2n}^{\leq s+t}$ [54, Theorem 1].

2.4 The Cohn–Elkies bound for sphere packings

As briefly mentioned in Chapter 1, the main ingredient of Viazovska's and Cohn et al.'s proofs of the optimality of E_8 and Leech lattice sphere packings is the following linear programming bound for sphere packings by Cohn and Elkies [13].

Theorem 2.4.1 (Cohn–Elkies, Theorem 3.2 of [13])**.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be an admissible function satisfying the following three conditions for some $r > 0$:

1. $f(\mathbf{0}) = \widehat{f}(\mathbf{0}) > 0$;

2. $f(\mathbf{x}) \leq 0$ for $\|\mathbf{x}\| \geq r$;

3. $\widehat{f}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^d$.

Then the optimal density Δ_d of sphere packings in $\mathbb{R}^d$ is bounded above by

$$\Delta_d \leq \left(\frac{r}{2}\right)^d \frac{\pi^{d/2}}{(d/2)!}.$$

Hence one can prove the optimality of certain sphere packings by finding *magic functions* with the corresponding radius r. For example, when $d = 1$, the function

$$f(x) = \frac{1}{1-x^2}\left(\frac{\sin(\pi x)}{\pi x}\right)^2 \tag{2.59}$$

satisfies all the conditions above for $r = 1$, and yields the (trivial) bound $\Delta_1 \leq 1$. Based on their numerical experiments, Cohn and Elkies conjectured the existence of magic functions in dimensions 2, 8, and 24 giving the optimal bounds corresponding to the best known packings. More precisely, they used functions of the form $f(\mathbf{x}) = p(2\pi\|\mathbf{x}\|^2)e^{-\pi\|\mathbf{x}\|^2}$ where p is a polynomial, and used Laguerre polynomials to optimize p and obtain good upper bounds for Δ_d. In particular, for $d = 8$ and 24, they obtained the upper bounds

$$\Delta_8 \leq 1.000001 \cdot \Delta_{\mathrm{E}_8}$$
$$\Delta_{24} \leq 1.0007071 \cdot \Delta_{\Lambda_{24}}$$

where Δ_{E_8} and $\Delta_{\Lambda_{24}}$ are the densities of the E_8 and Leech lattices, respectively, and the latter was improved by Cohn and Kumar [15] to

$$\Delta_{24} \leq (1 + 1.65 \cdot 10^{-30}) \cdot \Delta_{\Lambda_{24}}$$

which gives strong evidence for the existence of magic functions in these dimensions.

2.5 Optimal sphere packings in dimensions 8 and 24 and quasimodular form inequalities

After a few years, Viazovska [80] found an elegant construction of a magic function in dimension 8 that proves the optimality of the E_8 lattice sphere packing, and it took only one more week for her and her colleagues to find a similar magic function in dimension 24 (for the Leech lattice) [19]. The existence of a magic function in dimension 2 is still wide open.

Viazovska's and Cohn et al.'s constructions are based on an ingenious use of quasimodular forms. They first decompose f as a sum of (±1)-Fourier eigenfunctions $f = f_+ + f_-$ (hence $\widehat{f_+} = f_+$ and $\widehat{f_-} = -f_-$), and assume that they have the following form:

$$f_\pm(\mathbf{x}) = \sin^2\left(\frac{\pi|\mathbf{x}|^2}{2}\right)\int_0^\infty \varphi_\pm(it)e^{-\pi|\mathbf{x}|^2 t}\mathrm{d}t$$

for $x \in \mathbb{R}^d$, where $\varphi_\pm$ is a function defined on $\mathbb{H}$. Here we can assume that f is radial (by averaging over spheres centered at the origin), and the $\sin^2$ factor is added to enforce roots at the desired radii, i.e. the lengths of the vectors in the E_8 or Leech lattices (if f is a magic function, then both f and $\widehat{f}$ should have zeros at the nonzero lattice points, which follows from the proof of Theorem 2.4.1). Surprisingly, under this assumption, they proved that $f_\pm$ is a (±1)-Fourier eigenfunction if and only if $\varphi_\pm$ behaves like a "modular form". For example, when $d = 8$, we have $\varphi_\pm(t) = t^2\psi_\pm(i/t)$ with[1]

$$\psi_+ = -\phi_0 = -\frac{(E_2E_4 - E_6)^2}{\Delta},$$
$$\psi_- = \frac{36}{\pi^2}\psi_S = -\frac{18}{\pi^2}\frac{\Theta_2^{12}(2\Theta_2^8 + 5\Theta_2^4\Theta_4^4 + 5\Theta_4^8)}{\Delta}.$$

The corresponding integrals only converge for $\|\mathbf{x}\| > \sqrt{2}$, and one needs to analytically continue to $0 \le \|\mathbf{x}\| \le \sqrt{2}$. (In fact, $f_\pm$ are originally defined as sums of four complex integrals over different contours, and the above form of the integral is obtained from functional equations for $\varphi_\pm$.) Under this construction, the nonpositivity and nonnegativity conditions $f(\mathbf{x}) \le 0$ and $\widehat{f}(\mathbf{x}) \ge 0$ reduce to

$$\psi_+(it) + \psi_-(it) < 0 \tag{2.60}$$
$$\psi_+(it) - \psi_-(it) > 0 \tag{2.61}$$

for all $t > 0$. These inequalities are unnatural in the sense that ψ_+ and ψ_- are quasimodular forms of different weights, 0 and -2 respectively, which makes it difficult to think of a conceptual reason why they should be true. Viazovska proved the inequalities by approximating the functions with Fourier expansions and reducing them to finite calculations, which can be checked by interval arithmetic with computer calculations. The $d = 24$ case is similar but more involved and requires careful computer analysis, relying on approximations of the Fourier expansions of the corresponding quasimodular forms and on Sturm's bound. The details can be found in Section 5.1.

[1] Here we normalized in a slightly different way. We have $f(\mathbf{0}) = \widehat{f}(\mathbf{0}) = \frac{5}{4\pi}$. This normalization will be also used in Section 5.3.

Chapter 3

Positive and completely positive quasimodular forms

3.1 Definitions and examples

In this section, we study properties of quasimodular forms that are positive on the imaginary axis or have nonnegative Fourier coefficients.

Definition 3.1.1. A quasimodular form $F \in \mathcal{QM}_w^s$ is *positive* if it takes positive real values on the (positive) imaginary axis ($F(it) > 0$ for all $t > 0$).

We denote by $\mathcal{QM}_w^{s,+} \subset \mathcal{QM}_w^s$ the subset of positive quasimodular forms.

Definition 3.1.2. A quasimodular form F is *completely positive* if its Fourier coefficients are real and nonnegative, i.e. $F(z) = \sum_{n \geq n_0} a_n q^n$ with $a_n \geq 0$ for all $n \geq n_0$.

We denote the set of such forms of weight w and depth s as $\mathcal{QM}_w^{s,++}$. Both $\mathcal{QM}_w^{s,+}$ and $\mathcal{QM}_w^{s,++}$ are *convex cones* in $\mathcal{QM}_w^s$: they are closed under positive linear combinations. Clearly we have $\mathcal{QM}_w^{s,++} \subset \mathcal{QM}_w^{s,+}$, and the inclusion is strict in general. For example, $\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24} \in \mathcal{QM}_{12}^{0,+}$ but $\Delta(z) = q - 24q^2 + 252q^3 - 1472q^4 + \cdots \notin \mathcal{QM}_{12}^{0,++}$.

3.2 Derivative and positivity

Differentiation preserves complete positivity of quasimodular cusp forms, since it changes the n-th Fourier coefficient from a_n to na_n.

Proposition 3.2.1. Let $F \in \mathcal{QM}_w^s$ and $F' \in \mathcal{QM}_{w+2}^{s+1}$. Assume F is a cusp form. Then $F \in \mathcal{QM}_w^{s,++}$ if and only if $F' \in \mathcal{QM}_{w+2}^{s+1,++}$.

Positivity (not necessarily complete) is not preserved under derivatives in general. For example, the discriminant form Δ is positive due to its product expansion, but its derivative $\Delta' = E_2\Delta$ has a unique simple zero on the imaginary axis ($\lim_{t\to\infty} E_2(it) = 1$ and $\lim_{t\to 0^+} E_2(it) = -\infty$). However, positivity is preserved under *antiderivatives.*

Proposition 3.2.2. Let $X \in \mathcal{QM}_w^s$ be a cusp form. If $X' \in \mathcal{QM}_{w+2}^{s+1,+}$, then $X \in \mathcal{QM}_w^{s,+}$.

Proof. Let $x(t) := X(it)$ for $t > 0$. If $X' \in \mathcal{QM}_{w+2}^{s+1,+}$, then $\frac{\mathrm{d}x}{\mathrm{d}t} = -2\pi X'(it) < 0$ and $x(t)$ is strictly decreasing for all t. Hence $x(t) > \lim_{u\to\infty} x(u) = 0$. □

Complete positivity can be characterized as positivity of (higher) derivatives. To prove this, we need the following version of Bernstein's theorem.[1]

Theorem 3.2.3 (Bernstein)**.** Let $g : (0,\infty) \to \mathbb{R}$ be a smooth function. Then the following are equivalent:

1. $g(t)$ is a *completely monotone function*, i.e. $(-1)^n g^{(n)}(t) \geq 0$ for all $n \geq 0$ and $t > 0$.
2. There exists a unique nonnegative measure μ on $(0,\infty)$ such that

$$g(t) = \int_0^\infty e^{-tu}\,\mathrm{d}\mu(u)$$

where the integral converges absolutely.

A proof can be found in [68, Theorem 4.8, page 40]. Note that the original version of Bernstein's theorem considers functions on $[0,\infty)$ and finite measures, and this is a slightly generalized version of it.

Proposition 3.2.4. A cusp form $F \in \mathcal{QM}_w^s$ is completely positive if and only if all of its derivatives are positive.

[1] This theorem was the motivation for the naming of *complete* positivity.

Proof. This is a direct corollary of Theorem 3.2.3. Observe that $\frac{\mathrm{d}^k}{\mathrm{d}t^k}F(it) = (-2\pi)^k F^{(k)}(it)$, and the series $f(t) = \sum_{n\geq n_0} a_n e^{-2\pi n t}$ is the Laplace transform of the measure $\mu = \sum_{n\geq n_0} a_n \delta_{2\pi n}$. Although the measure μ is not finite, the integral

$$f(t) = \int_0^\infty e^{-tu}\mathrm{d}\mu(u)$$

converges absolutely (by the polynomial growth of the coefficients [69, Theorem 5, page 94]) and we can apply Theorem 3.2.3. Note that μ is supported on $(0,\infty)$ since F is a cusp form. □

3.3 Serre derivative and positivity

We can also prove that the *anti*-Serre derivative preserves positivity, which is a simple but surprisingly powerful result.

Proposition 3.3.1. Let $F \in \mathcal{QM}_w^s$ be a quasimodular form. Assume that there exists k and $t_0 > 0$ such that $(\partial_k F)(it) > 0$ for all $0 < t < t_0$ and $F(it_0) > 0$. Then $F(it) > 0$ for all $0 < t \leq t_0$.

Proof. Using $\Delta' = E_2\Delta$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{F(it)}{\Delta(it)^{\frac{k}{12}}}\right) = (-2\pi)\frac{F'(it)\Delta(it)^{\frac{k}{12}} - F(it)\frac{k}{12}E_2(it)\Delta(it)^{\frac{k}{12}}}{\Delta(it)^{\frac{k}{6}}} = (-2\pi)\frac{(\partial_k F)(it)}{\Delta(it)^{\frac{k}{12}}} < 0,$$

hence $t \mapsto F(it)/\Delta(it)^{\frac{k}{12}}$ is monotone decreasing and the result follows. □

Corollary 3.3.2. Let $F \in \mathcal{QM}_w^s$ be a quasimodular form. If $\partial_k F \in \mathcal{QM}_{w+2}^{s+1,+}$ and $F(it) > 0$ for *sufficiently large* $t > 0$, then $F \in \mathcal{QM}_w^{s,+}$. In particular, the assumption holds if all the Fourier coefficients of F are real and the first nonzero Fourier coefficient of F is positive.

Proof. The first assertion directly follows from Proposition 3.3.1. For the last assertion, when F has a Fourier expansion $F(z) = \sum_{n\geq n_0} a_n q^n$ with $a_{n_0} > 0$, then

$$e^{2\pi n_0 t}F(it) = a_{n_0} + e^{-2\pi t}\sum_{n\geq n_0+1} a_n e^{-2\pi(n-n_0-1)t}$$

and $\lim_{t\to\infty} e^{2\pi n_0 t}F(it) = a_{n_0} > 0$, so $F(it) > 0$ for sufficiently large t. □

Remark 3.3.3. It is also possible to *solve* the differential equation $\partial_k F = G$ and express $f(t) = F(it)$ in terms of $g(t) = G(it)$ as

$$f(t) = \left(\frac{\eta(it)}{\eta(it_0)}\right)^{2k} f(t_0) + 2\pi \int_t^{t_0} \left(\frac{\eta(it)}{\eta(iu)}\right)^{2k} g(u)\mathrm{d}u. \tag{3.1}$$

Moreover, Proposition 3.3.1 holds for a more general class of functions, for example, functions differentiable on $(0,\infty)$ with

$$\partial_k := D - \frac{k}{12}E_2(it) = -\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t} - \frac{k}{12}E_2(it),$$

and this version is used in the proof of inequality (5.9).

In general, the Serre derivative does not preserve complete positivity, e.g., E_4 is completely positive but $\partial_4 E_4 = -\frac{E_6}{3} = -\frac{1}{3} + 168q + \cdots$ is not. However, when the vanishing order at the cusp is sufficiently large, then it actually does.

Proposition 3.3.4. Let $F = \sum_{n\geq n_0} a_n q^n \in \mathcal{QM}_w^{s,++}$. For $k \geq 0$ and $n \geq k/12$, the n-th coefficient of $\partial_k F$ is nonnegative. In particular, if $n_0 \geq k/12 \geq 0$, $\partial_k F$ is also completely positive.

Proof. One can directly check that the Fourier expansion of the Serre derivative is

$$\begin{aligned}
\partial_k F &= \partial_k \left(a_{n_0}q^{n_0} + a_{n_0+1}q^{n_0+1} + a_{n_0+2}q^{n_0+2} + \cdots\right) \\
&= \left(n_0 a_{n_0} q^{n_0} + (n_0+1)a_{n_0+1}q^{n_0+1} + \cdots\right) \\
&\quad - \frac{k}{12}(1 - 24q - 72q^2 - 96q^3 - \cdots)\left(a_{n_0}q^{n_0} + a_{n_0+1}q^{n_0+1} + a_{n_0+2}q^{n_0+2} + \cdots\right) \\
&= \left(n_0 - \frac{k}{12}\right) a_{n_0} q^{n_0} + \left(\left(n_0 + 1 - \frac{k}{12}\right) a_{n_0+1} + 2k a_{n_0}\right) q^{n_0+1} + \cdots \\
&\quad + \left(\left(n_0 + m - \frac{k}{12}\right) a_{n_0+m} + 2k \sum_{j=1}^{m} \sigma_1(m+1-j) a_{n_0+j-1}\right) q^{n_0+m} + \cdots .
\end{aligned}$$

Hence if $n_0 \geq k/12$ and $a_j \geq 0$ for all $j \geq n_0$, the Fourier coefficients of $\partial_k F$ are also all nonnegative. □

3.4 Level and positivity

We also consider (completely) positive quasimodular forms of higher level. For completely positive forms, we consider only the q-expansions at the cusp $i\infty$, although a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ has several cusps in general. One easy way to construct (completely) positive quasimodular forms of level $\Gamma_0(N)$ is by using *old forms*.

Proposition 3.4.1. Let $F(z) \in \mathcal{QM}_w^s(\mathrm{SL}_2(\mathbb{Z}))$ be a positive (resp. completely positive) quasimodular form of weight w and depth s. Then for any $N \in \mathbb{Z}_{\geq 1}$, the form $G(z) := F(Nz) \in \mathcal{QM}_w^s(\Gamma_0(N))$ is also a positive (resp. completely positive) quasimodular form.

Proof. If F has a q-expansion $F(z) = \sum_{n \geq n_0} a_n q^n$, then $G(it) = F(iNt)$ and $G(z) = \sum_{n \geq n_0} a_n q^{Nn}$, and the proposition immediately follows. □

When F' is positive, we can subtract $F(Nz)$ from $F(z)$ and still maintain positivity.

Proposition 3.4.2. Let $F \in \mathcal{QM}_w^s(\mathrm{SL}_2(\mathbb{Z}))$. If F' is positive, then the form $F(z) - F(Nz) \in \mathcal{QM}_w^s(\Gamma_0(N))$ is also positive.

Proof. By $F'(it) = -\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}F(it)$, positivity of F' implies that the function $t \mapsto F(it)$ is monotone decreasing, hence the proposition follows. □

It may be possible to subtract larger multiples of $F(Nz)$ and still maintain positivity, or even complete positivity. For example, when $F(z) = -E_2'(z)/24 = \sum_{n \geq 1} n\sigma_1(n)q^n$, $F(z) - 4F(2z)$ is still completely positive since

$$n\sigma_1(n) - 4 \cdot \frac{n}{2}\sigma_1\left(\frac{n}{2}\right) = n\left(\sigma_1(n) - 2\sigma_1\left(\frac{n}{2}\right)\right) = n\sigma_1(m) > 0,$$

where $n = 2^k m$ with m odd. Also, one can show that $F(z) - 8F(2z)$ is positive, where 8 is the optimal constant. Such results can be proven by considering inequalities involving both modular forms and polynomials in t, which is the theme of Chapter 6.

Chapter 4

Positivity of Extremal Quasimodular Forms

In this chapter, we study the (complete) positivity of Kaneko and Koike's extremal quasimodular forms [40]. In particular, we prove their conjecture on complete positivity for depth 1 extremal quasimodular forms (Corollary 4.2.2), and a weak version of the conjecture for depth 2 (Theorem 4.4.4).

4.1 Extremal quasimodular forms à la Kaneko and Koike

In [40], Kaneko and Koike introduced and studied *extremal quasimodular forms* of level 1, defined as follows.

Definition 4.1.1 (Kaneko–Koike [40])**.** For a given weight w and depth s, a quasimodular form $f \in \mathcal{QM}_w^s \backslash \mathcal{QM}_w^{s-1}$ is *extremal* if, for $m = \dim_{\mathbb{C}} \mathcal{QM}_w^s$, the first m Fourier coefficients of $f = \sum_{n\geq 0} a_n q^n$ are

$$a_0 = a_1 = \cdots = a_{m-2} = 0, a_{m-1} \neq 0.$$

f is called *normalized* if $a_{m-1} = 1$.

Kaneko and Koike conjectured the existence and uniqueness of (normalized) extremal forms for each (even) weight w and depth s (satisfying $0 \leq s \leq w/2, s \neq \frac{w}{2} - 1$), and gave explicit examples for depths 1 and 2 defined by recursive formulas satisfying certain modular linear differential equations. Pellarin [61] established uniqueness for $s \leq 4$, and Grabner [32] extended Kaneko–Koike's result and constructed differential equations satisfied by depth ≤ 4 extremal quasimodular forms. For these depths, we will denote the normalized (i.e., the first nonzero Fourier coefficient is one) extremal form of weight w and depth s by $X_{w,s}$. For each w satisfying a certain congruence condition depending on s, $X_{w,s}$ satisfies the following modular linear differential equation:

$$\theta^{(2)}_{w-2} X_{w,s} = X^{(s+1)}_{w,s} - \frac{w}{12}[E_2, X_{w,s}]^{(2,w-s)}_s = 0. \tag{4.1}$$

Here the operator

$$\theta^{(r)}_k = D^{r+1} - \frac{k+r}{12}[E_2, -]^{(2,k)}_r \tag{4.2}$$

is the *Kaneko–Zagier operator* [43, 40] and $[-,-]$ is the Rankin–Cohen bracket (2.58). Kaneko and Koike also conjectured that the Fourier coefficients of extremal forms of depth ≤ 4 are all positive [40, Conjecture 2], and Grabner [31] proved the conjecture *for all but finitely many coefficients*. The proof uses Jenkins and Rouse's explicit version of Deligne's bound [38] (see Theorem 9.4.2 for the precise statement).

Here are the simplest examples:

Lemma 4.1.2. The following quasimodular forms are extremal and completely positive:

$$\begin{aligned} X_{4,2} &= \frac{1}{288}(E_4 - E_2^2) \in \mathcal{QM}^{2,++}_4, \\ X_{6,1} &= \frac{1}{720}(E_2E_4 - E_6) \in \mathcal{QM}^{1,++}_6, \\ X_{8,1} &= \frac{1}{1008}(E_4^2 - E_2E_6) \in \mathcal{QM}^{1,++}_8. \end{aligned}$$

Proof. Extremality is verified in [40]. Complete positivity follows directly from Ramanujan's identities,

$$\begin{aligned} X_{4,2} &= \frac{1}{288}(E_4 - E_2^2) = -\frac{1}{24}E_2' = \sum_{n\geq 1} n\sigma_1(n)q^n, \\ X_{6,1} &= \frac{1}{720}(E_2E_4 - E_6) = \frac{1}{240}E_4' = \sum_{n\geq 1} n\sigma_3(n)q^n, \\ X_{8,1} &= \frac{1}{1008}(E_4^2 - E_2E_6) = -\frac{1}{504}E_6' = \sum_{n\geq 1} n\sigma_5(n)q^n. \end{aligned}$$

□

4.2 Kaneko–Koike's conjecture for depth 1 extremal quasimodular forms

In this section, we prove the conjecture of Kaneko and Koike [40, Conjecture 2] in the case of depth 1 (Corollary 4.2.2). The main idea is to prove new recurrence relations (Theorem 4.2.1) and use induction on weights.

For even $w \geq 6$, let $X_w = X_{w,1}$ be the unique normalized extremal quasimodular form of weight w and depth 1. We have $X_6 = (E_2E_4 - E_6)/720$, and the forms X_w satisfy the following recursive formula for $w \geq 6$ with $w \equiv 0 \pmod 6$ [40, 32]:

$$X_{w+2} = \frac{12}{w+1}\partial_{w-1}X_w \tag{4.3}$$

$$X_{w+4} = E_4X_w \tag{4.4}$$

$$\begin{aligned} X_{w+6} &= \frac{w+6}{72(w+1)(w+5)}\left(E_4\partial_{w-1}X_w - \frac{w+1}{12}E_6X_w\right) \\ &= \frac{w+6}{864(w+5)}\left(E_4X_{w+2} - E_6X_w\right). \end{aligned} \tag{4.5}$$

The vanishing order of X_w at the cusp is $\lfloor \frac{w}{6} \rfloor$. Moreover, when $w \equiv 0 \pmod 6$, X_w is a solution of the differential equation

$$\theta_{w-1}^{(1)}X_w = X_w'' - \frac{w}{6}E_2X_w' + \frac{w(w-1)}{12}E_2'X_w = 0. \tag{4.6}$$

We prove a new recurrence relation (4.7), which immediately implies the complete positivity of the forms X_w.

Theorem 4.2.1. Let $X_w = X_{w,1}$ be the unique normalized extremal quasimodular form of weight w and depth 1. For $w \equiv 0 \pmod 6$ and $w \geq 12$, we have

$$X_w' = \frac{5w}{72}X_6X_{w-4} + \frac{7w}{72}X_8X_{w-6}. \tag{4.7}$$

$$X_{w+2}' = \frac{5w}{72}X_6X_{w-2} + \frac{7w}{72}X_8X_{w-4}, \tag{4.8}$$

$$X_{w+4}' = 240X_6X_w + \frac{7w}{72}X_8X_{w-2} + \frac{5w}{72}X_{10}X_{w-4} \tag{4.9}$$

Proof. $(4.3)_{w-4}$ is equivalent to

$$X_{w-4}' = \frac{w-5}{12}E_2X_{w-4} + \frac{w-7}{12}E_4X_{w-6} \tag{4.10}$$

and differentiating $(4.5)_{w-6}$ gives

$$\begin{aligned}
X'_w &= \frac{w}{864(w-1)}(E'_4X_{w-4} + E_4X'_{w-4} - E'_6X_{w-6} - E_6X'_{w-6}) && \cdots(4.5)_{w-6}\\
&= \frac{w}{864(w-1)}\left(\frac{E_2E_4 - E_6}{3}X_{w-4} + \frac{w-5}{12}E_2E_4X_{w-4} + \frac{w-7}{12}E_4^2X_{w-6}\right. && \cdots(4.10)_w\\
&\quad \left.-\frac{E_2E_6 - E_4^2}{2}X_{w-6} - E_6\left(\frac{w-5}{12}X_{w-4} + \frac{w-7}{12}E_2X_{w-6}\right)\right) && \cdots(4.3)_{w-6}\\
&= \frac{w}{864(w-1)}\left(\frac{w-1}{12}(E_2E_4 - E_6)X_{w-4} + \frac{w-1}{12}(E_4^2 - E_2E_6)X_{w-6}\right)\\
&= \frac{5w}{72}X_6X_{w-4} + \frac{7w}{72}X_8X_{w-6}.
\end{aligned}$$

(4.8) and (4.9) can be proved similarly, and we omit the proof. □

Corollary 4.2.2. Kaneko–Koike's conjecture holds for depth 1 extremal forms.

Proof. The conjecture holds for X_6 and X_8 by Lemma 4.1.2, and $X_{10} = E_4X_6$ shows that X_{10} is also completely positive. Now, assume that X_k is completely positive for $k \le w-2$. Then $(4.7)_w$ implies that X_w is also completely positive. Combining this with Proposition 3.3.4 (recall that the vanishing order of X_w at the cusp is $\frac{w}{6} > \frac{w-1}{12}$) shows that $X_{w+2} \in \mathcal{QM}^{1,++}_{w+2}$. We also get $X_{w+4} \in \mathcal{QM}^{1,++}_{w+4}$ from (4.4), since E_4 has nonnegative Fourier coefficients. □

Remark 4.2.3. In fact, we can prove the Kaneko–Koike conjecture for depth 1 extremal forms using only the original recurrence relations (4.3)–(4.5) and induction, without using the new recurrence relations in Theorem 4.2.1. Assume that X_w is completely positive for some $w \equiv 0 \pmod 6$. Then the complete positivity of X_{w+2} and X_{w+4} follows as before. For X_{w+6}, we can rewrite (4.5) as

$$X_{w+6} = \frac{w+6}{864(w+5)}\left((E_4-1)X_{w+2} + (1-E_6)X_w + (X_{w+2} - X_w)\right)$$

and since $E_4 - 1$ and $1 - E_6$ are completely positive, it is enough to show that $X_{w+2} - X_w$ is completely positive. Using (4.3), we have

$$X_{w+2} - X_w = \frac{12}{w+1}X'_w - \frac{w-1}{w+1}E_2X_w - X_w = \frac{w-1}{w+1}(1-E_2)X_w + \frac{12}{w+1}\left(X'_w - \frac{w}{6}X_w\right).$$

Since the vanishing order of X_w at the cusp is $\frac{w}{6}$, we can write the Fourier expansion of X_w as $X_w = q^{w/6} + \sum_{n \ge \frac{w}{6}+1} a_nq^n$ with $a_n \ge 0$. Then

$$X'_w - \frac{w}{6}X_w = \sum_{n \ge \frac{w}{6}+1}\left(n - \frac{w}{6}\right)a_nq^n$$

is also completely positive, which completes the induction step.

4.3 Modular components of depth 1 extremal quasimodular forms

Each depth 1 extremal quasimodular form $X_w = X_{w,1}$ can be written as

$$X_w = A_w + E_2 B_{w-2} \tag{4.11}$$

where A_w, B_{w-2} are modular forms of weight w and $w-2$, respectively. In this section, we study the constant terms of A_w and B_{w-2}, which will be used in Chapter 6. Using the recurrence relations (4.3)–(4.5), we can derive the following recurrence relations for A_w and B_{w-2}:

Proposition 4.3.1. For $6 \mid w$ and $w \geq 12$, we have

$$A_{w+2} = \frac{12}{w+1}\left(\partial_w A_w - \frac{1}{12}E_4 B_{w-2}\right) \tag{4.12}$$

$$A_{w+4} = E_4 A_w \tag{4.13}$$

$$A_{w+6} = \frac{w+6}{864(w+5)}(E_4 A_{w+2} - E_6 A_w) \tag{4.14}$$

$$B_w = \frac{12}{w+1}\left(\frac{1}{12}A_w + \partial_{w-2} B_{w-2}\right) \tag{4.15}$$

$$B_{w+2} = E_4 B_w \tag{4.16}$$

$$B_{w+4} = \frac{w+6}{864(w+5)}(E_4 B_{w+2} - E_6 B_w). \tag{4.17}$$

Proof. From (4.3), we have

$$\begin{aligned} X_{w+2,1} &= A_{w+2} + E_2 B_w \qquad (4.18)\\ &= \frac{12}{w+1}\left(X'_{w,1} - \frac{w-1}{12}E_2 X_{w,1}\right)\\ &= \frac{12}{w+1}\left(A'_w + E_2 B'_{w-2} - \frac{w-1}{12}E_2(A_w + E_2 B_{w-2})\right)\\ &= \frac{12}{w+1}\left(\partial_w A_w + \frac{w}{12}E_2 A_w + E_2\left(\partial_{w-2}B_{w-2} + \frac{w-2}{12}E_2 B_{w-2}\right) + \frac{E_2^2 - E_4}{12}B_{w-2}\right.\end{aligned}$$

$$-\frac{w-1}{12}E_2A_w-\frac{w-1}{12}E_2^2B_{w-2}\Big)$$

$$=\frac{12}{w+1}\left(\left(\partial_w A_w-\frac{1}{12}E_4B_{w-2}\right)+E_2\left(\partial_{w-2}B_{w-2}+\frac{1}{12}A_w\right)\right) \tag{4.19}$$

and comparing (4.18) and (4.19) proves (4.12) and (4.15). The other relations follow similarly from (4.4) and (4.5). □

From these recurrence relations, we obtain recurrence relations for the Fourier expansions

$$A_w=\sum_{n\geq 0}\alpha_{w,n}q^n,\quad B_{w-2}=\sum_{n\geq 0}\beta_{w-2,n}q^n. \tag{4.20}$$

Lemma 4.3.2. For $w\equiv 0 \pmod 6$, the constant terms $\alpha_{w,0}$ and $\beta_{w-2,0}$ satisfy

$$\alpha_{w+2,0}=-\frac{w-1}{w+1}\alpha_{w,0} \tag{4.21}$$

$$\alpha_{w+4,0}=\alpha_{w,0} \tag{4.22}$$

$$\alpha_{w+6,0}=-\frac{w(w+6)}{432(w+1)(w+5)}\alpha_{w,0} \tag{4.23}$$

and $\beta_{w,0}=-\alpha_{w+2,0}$ for all $w\geq 4$. In particular, for all $w\geq 6$, neither A_w nor B_{w-2} is a cusp form. Also, for $w\equiv 0 \pmod 6$, we have

$$\alpha_w=(-1)^{\frac{w}{6}}\frac{(w/6)!(w/3)!(w/2)!}{2w\cdot w!}. \tag{4.24}$$

Proof. Since $X_{w,1}=A_w+E_2B_{w-2}=O(q)$, we have $\beta_{w-2}=-\alpha_w$ by comparing the constant terms. For $w\equiv 0 \pmod 6$, by comparing the constant terms in (4.12), we have

$$\alpha_{w+2}=\frac{12}{w+1}\left(-\frac{w}{12}\alpha_w-\frac{1}{12}\beta_{w-2}\right)=-\frac{w-1}{w+1}\alpha_w.$$

The other two recurrence relations (4.22) and (4.23) can be proved similarly using (4.13) and (4.14). From $X_{6,1}=\frac{E_2E_4-E_6}{720}$, we have $\alpha_6=-\frac{1}{720}$ and $\beta_4=\frac{1}{720}$, so α_w,β_w are nonzero and A_w, B_{w-2} are not cusp forms. Finally, the closed formula for α_w follows from (4.23). □

Lemma 4.3.3. For $w\geq 12$ with $6\mid w$, we have

$$\frac{\alpha_{w,1}}{\alpha_{w,0}}=-\frac{12(w-3)(w+4)}{w-6} \tag{4.25}$$

$$\frac{\alpha_{w+2,1}}{\alpha_{w+2,0}} = -\frac{12(w^2-9w-24)}{w-6} \tag{4.26}$$

$$\frac{\alpha_{w+4,1}}{\alpha_{w+4,0}} = -\frac{12(w^2-19w+108)}{w-6} \tag{4.27}$$

$$\frac{\beta_{w-2,1}}{\beta_{w-2,0}} = -\frac{12(w-1)w}{w-6} \tag{4.28}$$

$$\frac{\beta_{w,1}}{\beta_{w,0}} = -\frac{12(w-12)(w+1)}{w-6} \tag{4.29}$$

$$\frac{\beta_{w+2,1}}{\beta_{w+2,0}} = -\frac{12(w^2-21w+120)}{w-6} \tag{4.30}$$

Proof. Use induction on w. The base case $w = 12$ can be directly checked from

$$\begin{aligned}
X_{12,1} &= \frac{-12E_2E_4E_6 + 5E_4^3 + 7E_6^2}{3991680} = A_{12} + E_2B_{10},\\
A_{12} &= \frac{5E_4^3 + 7E_6^2}{3991680} = \frac{1}{332640} - \frac{1}{1155}q + O(q^2), \quad B_{10} = -\frac{E_4E_6}{332640} = -\frac{1}{332640} + \frac{1}{1260}q + O(q^2),\\
X_{14,1} &= \frac{7E_2E_4^3 + 5E_2E_6^2 - 12E_4^2E_6}{4717440} = A_{14} + E_2B_{12},\\
A_{14} &= -\frac{E_4^2E_6}{391320} = -\frac{1}{391320} + \frac{1}{16380}q + O(q^2), \quad B_{12} = \frac{7E_4^3 + 5E_6^2}{4717440} = \frac{1}{393120} + O(q^2),\\
X_{16,1} &= \frac{E_4(-12E_2E_4E_6 + 5E_4^2 + 7E_6^2)}{3991680} = A_{16} + E_2B_{14},\\
A_{16} &= \frac{E_4(5E_4^2 + 7E_6^2)}{3991680} = \frac{1}{332640} - \frac{1}{6930}q + O(q^2), \quad B_{14} = -\frac{E_4^2E_6}{332640} = -\frac{1}{332640} + \frac{1}{13860}q + O(q^2).
\end{aligned}$$

By comparing the coefficients of q in (4.12), we have

$$\alpha_{w+2,1} = \frac{12}{w+1}\left(2(w+10)\alpha_{w,0} + \left(-\frac{w}{12}+1\right)\alpha_{w,1} - \frac{1}{12}\beta_{w-2,1}\right)$$

and thus

$$\begin{aligned}
\frac{\alpha_{w+2,1}}{\alpha_{w+2,0}} &= \frac{12}{w+1}\left(2(w+10)\cdot\frac{\alpha_{w,0}}{\alpha_{w+2,0}} + \left(-\frac{w}{12}+1\right)\cdot\frac{\alpha_{w,0}}{\alpha_{w+2,0}}\frac{\alpha_{w,1}}{\alpha_{w,0}} + \frac{1}{12}\cdot\frac{\alpha_{w,0}}{\alpha_{w+2,0}}\frac{\beta_{w-2,1}}{\beta_{w-2,0}}\right)\\
&= -\frac{12}{w-1}\left(2(w+10) + \left(-\frac{w}{12}+1\right)\cdot\frac{-12(w-3)(w+4)}{w-6} + \frac{1}{12}\cdot -\frac{12(w-1)w}{w-6}\right)\\
&= -\frac{12(w^2-9w-24)}{w-6}.
\end{aligned}$$

The other formulas can be proved similarly. □

4.4 Kaneko–Koike's conjecture for depth 2 extremal forms

For even $w \geq 4$ with $w \equiv 0 \pmod 4$, the depth 2 (normalized) extremal forms $X_{w,2}$ satisfy the following recurrence relations [32][1]: $X_{4,2} = \frac{E_4 - E_2^2}{288} = -\frac{E_2'}{24}$ and

$$X_{w+4,2} = \frac{3(w+4)^2}{16(w+1)(w+2)^2(w+3)} \left(\frac{w(w+1)}{36} E_4 X_{w,2} - \partial_{w-2}^2 X_{w,2} \right) \tag{4.31}$$

$$X_{w+2,2} = \frac{6}{w+1} \partial_{w-2} X_{w,2}. \tag{4.32}$$

$$= \frac{3w^2}{16(w^2-1)(w-6)^2} \left(\frac{(w-4)(w-5)}{36} E_4 X_{w-2,2} - \partial_{w-4}^2 X_{w-2,2} \right) \tag{4.33}$$

The vanishing order of $X_{w,2}$ at the cusp is $\lfloor \frac{w}{4} \rfloor$. Also, $X_{w,2}$ (for $w \equiv 0 \pmod 4$) is a solution of the differential equation

$$\theta_{w-2}^{(2)} X_{w,2} = X_{w,2}''' - \frac{w}{12} [E_2, X_{w,2}]_2^{(2,w-2)} \tag{4.34}$$

$$= X_{w,2}''' - \frac{w}{4} E_2 X_{w,2}'' + \frac{w(w-1)}{4} E_2' X_{w,2}' - \frac{w(w-1)(w-2)}{24} E_2'' X_{w,2} \tag{4.35}$$

$$= \partial_{w-2}^3 X_{w,2} - \frac{3w^2-4}{144} E_4 \partial_{w-2} X_{w,2} - \frac{(w-2)^2(w+1)}{864} E_6 X_{w,2} = 0. \tag{4.36}$$

The forms $X_{w,2}$ also satisfy the following recurrence relation:

Proposition 4.4.1. For each $w \geq 12$ that is a multiple of 4, we have

$$X_{w+2,2} = \frac{w^2}{768(w-1)(w+1)} (E_4 X_{w-2,2} - E_6 X_{w-4,2}) \tag{4.37}$$

Proof. By applying ∂_{w-2} to $(4.31)_{w-4}$ and using $(4.32)_w$, we can obtain an expression for $X_{w+2,2}$ as a combination of $\partial_{w-2}(E_4 X_{w-4,2})$ and $\partial_{w-6}^3 X_{w-4,2}$. Now use $(4.35)_{w-4}$ to substitute the $\partial_{w-6}^3 X_{w-4,2}$ term. This proves $(4.37)_w$. □

For small weights, the following exceptional identities establish complete positivity of the depth 2 extremal forms.

[1] There is a minor error in [26]. We need to replace w^2 with $(w+4)^2$ in the numerator to make $X_{w+4,2}$ normalized. We correct this in (4.31).

Proposition 4.4.2. We have the following identities:

$$X'_{8,2} = 2X_{4,2}X_{6,1}, \tag{4.38}$$

$$X'_{10,2} = \frac{8}{9}X_{4,2}X_{8,1} + \frac{10}{9}X_{6,1}^2, \tag{4.39}$$

$$X'_{12,2} = 3X_{6,1}X_{8,2}, \tag{4.40}$$

$$X'_{14,2} = 3X_{4,2}X_{12,1}. \tag{4.41}$$

In particular, $X_{w,2}$ is completely positive for $w \leq 14$.

Proof. Complete positivity follows from the identities and Proposition 3.2.1. □

Unfortunately, we do not know how to prove complete positivity of $X_{w,2}$ for general w. However, we can prove a weaker version, namely positivity of $X_{w,2}$. This is based on Nakaya's hypergeometric expressions for $X_{w,2}$ [57] and the recurrence relations established above.

Let ${}_3F_2$ be the hypergeometric function

$${}_3F_2(a_1, a_2, a_3; b_1, b_2; t) = \sum_{n\geq 0} \frac{(a_1)_n(a_2)_n(a_3)_n}{(b_1)_n(b_2)_n}\frac{t^n}{n!}. \tag{4.42}$$

Nakaya proved that $X_{w,2}$ admits the following hypergeometric expressions.

Theorem 4.4.3 (Nakaya, Proposition 6.1 of [57])**.** Normalized extremal quasimodular forms $X_{w,2}$ for $w \geq 4$ and $w \neq 6$ can be expressed as hypergeometric series:

$$X_{4k,2}(z) = j(z)^{-k}E_4(z)^{\frac{2k-1}{2}} \cdot {}_3F_2\left(\frac{4k+1}{6}, \frac{4k+3}{6}, \frac{4k+5}{6}; k+1, k+1; \frac{1728}{j(z)}\right), \tag{4.43}$$

$$X_{4k+2,2}(z) = j(z)^{-k}E_4(z)^{\frac{2k-3}{2}}E_6(z) \cdot {}_3F_2\left(\frac{4k+3}{6}, \frac{4k+5}{6}, \frac{4k+7}{6}; k+1, k+1; \frac{1728}{j(z)}\right), \tag{4.44}$$

when $z = it$ and $t \geq 1$.

Proof. For the sake of completeness, we include the details of the proof, which are omitted in [57]. When $w = 4k$, (4.43) can be shown by proving that both sides satisfy the same differential equation (4.35) and initial conditions, as suggested in [57, Section 6.1]. More precisely, from (4.35) $g(z) = E_4(z)^{-\frac{w-2}{4}}X_{w,2}(z)$ with $x = 1728/j(z)$ satisfies the equation

$$x^2(1-x)g'''(x) + x\left(-\frac{w-12}{4} + \frac{w-18}{4}x\right)g''(x)$$

$$+\left(-\frac{w-4}{4}-\frac{3w^2-72w+452}{144}x\right)g'(x)+\frac{(w-10)(w-6)(w-2)}{1728}g(x)=0.$$

It has singularities at $x = 0, 1, \infty$, and the solution $g(x)$ analytically continues from $x = 0$ to $x = 1$, corresponding to $t = \infty$ to $t = 1$ for $z = it$.

For $X_{w+2,2} = X_{4k+2,2}$, we use (4.43) and (4.32):

$$\begin{aligned}
&\frac{w+1}{6}X_{w+2,2}\\
&=\partial_{w-2}X_{w,2}=X'_{w,2}-\frac{w-2}{12}E_2X_{w,2}\\
&=(j(z)^{-\frac{w}{4}}E_4(z)^{\frac{w-2}{4}})'\cdot {}_3F_2\left(\frac{w+1}{6},\frac{w+3}{6},\frac{w+5}{6};\frac{w}{4}+1,\frac{w}{4}+1;\frac{1728}{j(z)}\right)\\
&\quad+j(z)^{-\frac{w}{4}}E_4(z)^{\frac{w-2}{4}}\left(\frac{1728}{j(z)}\right)'\cdot\frac{\mathrm{d}}{\mathrm{d}x}{}_3F_2\left(\frac{w+1}{6},\frac{w+3}{6},\frac{w+5}{6};\frac{w}{4}+1,\frac{w}{4}+1;x\right)\bigg|_{x=1728/j(z)}\\
&\quad-\frac{w-2}{12}E_2(z)j(z)^{-\frac{w}{4}}E_4(z)^{\frac{w-2}{4}}\cdot{}_3F_2\left(\frac{w+1}{6},\frac{w+3}{6},\frac{w+5}{6};\frac{w}{4}+1,\frac{w}{4}+1;\frac{1728}{j(z)}\right)\\
&=\partial_{w-2}(j(z)^{-\frac{w}{4}}E_4(z)^{\frac{w-2}{4}})\cdot{}_3F_2\left(\frac{w+1}{6},\frac{w+3}{6},\frac{w+5}{6};\frac{w}{4}+1,\frac{w}{4}+1;\frac{1728}{j(z)}\right)\\
&\quad+j(z)^{-\frac{w}{4}}E_4(z)^{\frac{w-2}{4}}\left(\frac{1728}{j(z)}\right)'\cdot\frac{\mathrm{d}}{\mathrm{d}x}{}_3F_2\left(\frac{w+1}{6},\frac{w+3}{6},\frac{w+5}{6};\frac{w}{4}+1,\frac{w}{4}+1;x\right)\bigg|_{x=1728/j(z)}\\
&=\frac{w+1}{6}j(z)^{-\frac{w}{4}}E_4(z)^{\frac{w-6}{4}}E_6(z)\cdot{}_3F_2\left(\frac{w+1}{6},\frac{w+3}{6},\frac{w+5}{6};\frac{w}{4}+1,\frac{w}{4}+1;\frac{1728}{j(z)}\right)\\
&\quad+j(z)^{-\frac{w}{4}}E_4(z)^{\frac{w-6}{4}}E_6(z)\cdot x\frac{\mathrm{d}}{\mathrm{d}x}{}_3F_2\left(\frac{w+1}{6},\frac{w+3}{6},\frac{w+5}{6};\frac{w}{4}+1,\frac{w}{4}+1;x\right)\bigg|_{x=1728/j(z)}\\
&=\frac{w+1}{6}j(z)^{-\frac{w}{4}}E_4(z)^{\frac{w-6}{4}}E_6(z)\cdot{}_3F_2\left(\frac{w+7}{6},\frac{w+3}{6},\frac{w+5}{6};\frac{w}{4}+1,\frac{w}{4}+1;\frac{1728}{j(z)}\right)\\
&=\frac{w+1}{6}j(z)^{-\frac{w}{4}}E_4(z)^{\frac{w-6}{4}}E_6(z)\cdot{}_3F_2\left(\frac{w+3}{6},\frac{w+5}{6},\frac{w+7}{6};\frac{w}{4}+1,\frac{w}{4}+1;\frac{1728}{j(z)}\right).
\end{aligned}$$

The penultimate equality uses the identity

$$\left(1+\frac{x}{a_1}\frac{\mathrm{d}}{\mathrm{d}x}\right){}_3F_2(a_1,a_2,a_3;b_1,b_2;x)={}_3F_2(a_1+1,a_2,a_3;b_1,b_2;x).$$

□

Theorem 4.4.4. $X_{w,2}$ is *positive* for all $w \geq 4$.

Proof. We proceed by induction on w, treating the cases $t \geq 1$ and $0 < t < 1$ separately. The case $t \geq 1$ follows almost immediately from (4.43) and (4.44), since $E_4(it)$ and $j(it)$ are both positive. Now, we consider $0 < t < 1$. Assume that $X_{w-2,2}$ and $X_{w-4,2}$ are positive for $0 < t < 1$. Since $E_4(it) > 0$ and $E_6(it) < 0$ for $0 < t \leq 1$, (4.37) shows that $X_{w+2,2}$ is also positive on $0 < t \leq 1$. Now, Corollary 3.3.2, (4.32) and $X_{w,2}(i) > 0$ show that $X_{w,2}$ is positive for $0 < t < 1$. □

Remark 4.4.5. In [41], certain solutions of third-order analogues of Kaneko–Zagier modular differential equations [43] are characterized. In particular, solutions of *(vacuum) character type* are characterized; these are the solutions f for which the quotient f/η^{2k} has nonnegative integral Fourier coefficients, which implies positivity of f. However, our $X_{w,2}$ are not of this type since the coefficients are not integral except for $w = 4, 8$ [39].

4.5 Higher depth extremal forms

It is natural to ask whether we can prove similar results for higher depths. As mentioned above, we do not even have a uniqueness statement when the depth is greater than 4. However, for small weights it is easy to check uniqueness by testing whether the $d \times d$ matrix formed from the first d coefficients of a basis of $Q\mathcal{M}_w^{\leq s}(\mathrm{SL}_2(\mathbb{Z}))$ is invertible, where $d = \dim Q\mathcal{M}_w^{\leq s}(\mathrm{SL}_2(\mathbb{Z}))$. We checked this for depth ≤ 10 and weight ≤ 200, and we conjecture that uniqueness holds for all weights and depths (see Appendix A.2 for the Sage code).

Regarding (complete) positivity, we do not yet have any hypergeometric identities available for depth ≥ 3, so even the weak version of the conjecture, namely positivity of $X_{w,s}$ for $s \leq 4$, is still difficult. However, the naive extension of the Kaneko–Koike conjecture turns out to be false in higher depth. The smallest example we have found is $X_{16,5}$, whose q-expansion is

$$\begin{aligned} X_{16,5} &= -\frac{1}{3615458092646400} \\ &\quad \times (-1716715E_2^5E_6 + 3090087E_2^4E_4^2 - 2026934E_2^3E_4E_6 - 6076E_2^2E_4^3 + 798722E_2^2E_6^2 \\ &\qquad - 152175E_2E_4^2E_6 - 1208675E_4^4 + 1221766E_4E_6^2) \\ &= q^7 - \frac{4809}{16}q^8 - 3913q^9 - \frac{59577}{2}q^{10} - 166404q^{11} - 749910q^{12} - 2875670q^{13} \\ &\quad - 9712998q^{14} - 29591541q^{15} - \frac{661603971}{8}q^{16} - 214814775q^{17} - 523902498q^{18} + \cdots . \end{aligned}$$

In fact, we can prove that all the coefficients except the first one are negative, so $X_{16,5}$ is not even positive.

Proposition 4.5.1. All the q-coefficients of the extremal quasimodular form $X_{16,5}$ except the first one are negative. In particular, $X_{16,5}$ is not positive.

Proof. We can express $X_{16,5}$ as a linear combination

$$X_{16,5} = c_1 E_{14}' + c_2 E_{12}'' + c_3 E_{10}''' + c_4 E_8'''' + c_5 E_6''''' + c_6 \Delta E_4 + c_7 \Delta X_{4,2}$$

where

$$\begin{aligned} c_1 &= 1/4843238400 \\ c_2 &= 4531/4533271142400 \\ c_3 &= 149/60217344000 \\ c_4 &= 49/12317184000 \\ c_5 &= 7/1791590400 \\ c_6 &= 86619413/139015844352000 \\ c_7 &= -118801/10746432000, \end{aligned}$$

as can be checked directly in Sage. Using this, we can express the q-coefficients of $X_{16,5} = q^7 + \sum_{n\geq 8} a_n q^n$ in terms of divisor functions and their convolutions with the Ramanujan τ-function, namely

$$a_n = x_n + 240 c_6 y_n + c_7 z_n + c_6 \tau(n) \tag{4.45}$$

where

$$\begin{aligned} x_n &= -\frac{n\sigma_{13}(n)}{201801600} + \frac{4531 n^2 \sigma_{11}(n)}{47809681920} - \frac{149 n^3 \sigma_9(n)}{228096000} + \frac{49 n^4 \sigma_7(n)}{25660800} - \frac{49 n^5 \sigma_5(n)}{24883200} \\ y_n &= \sum_{k=1}^{n-1} \tau(k)\sigma_3(n-k) \\ z_n &= \sum_{k=1}^{n-1} \tau(k)(n-k)\sigma_1(n-k). \end{aligned}$$

Here $c_6\tau(n)$ is the contribution from ΔE_4.

The idea is to use the bounds on $\sigma_k(n)$ and $\tau(n)$ to show that $a_n < 0$ for sufficiently large n, and then check the remaining finitely many cases by computer. In particular, the $-n\sigma_{13}(n)$ term in x_n dominates the other terms for large n, and hence a_n is negative for sufficiently large n. With $n^k \le \sigma_k(n) < 2n^k$ for $n \ge 1$ and $k \ge 2$, we get an upper bound

$$x_n \le -\frac{n^{14}}{201801600} + \frac{9602n^{13}}{47809681920} + \frac{98n^{11}}{25660800} \tag{4.46}$$

for all $n \ge 1$. For y_n and z_n, we use Deligne's bound $|\tau(n)| \le \sigma_0(n)n^{\frac{11}{2}} \le n^{\frac{13}{2}}$ and $\sigma_1(n) \le n^2$ to bound them as

$$|y_n| \le \sum_{k=1}^{n-1} k^{\frac{13}{2}} 2(n-k)^3 \le n^4 \sum_{k=1}^{n-1} k^{\frac{13}{2}} \le 2n^3 \int_0^n x^{\frac{13}{2}}\,dx = \frac{4}{15}n^{\frac{21}{2}} \tag{4.47}$$

$$|z_n| \le \sum_{k=1}^{n-1} k^{\frac{13}{2}} (n-k)^3 \le n^3 \sum_{k=1}^{n-1} k^{\frac{13}{2}} \le n^3 \int_0^n x^{\frac{13}{2}}\,dx = \frac{2}{15}n^{\frac{21}{2}}. \tag{4.48}$$

Combining (4.46), (4.47), and (4.48), we have

$$\begin{aligned} a_n &\le -\frac{n^{14}}{201801600} + \frac{9602n^{13}}{47809681920} + \frac{98n^{11}}{25660800} + 240c_6 \cdot \frac{4}{15}n^{\frac{21}{2}} + c_7 \cdot \frac{2}{15}n^{\frac{21}{2}} + c_6 \cdot n^{\frac{13}{2}} \\ &\le -\frac{n^{14}}{201801600} + \frac{9602n^{13}}{47809681920} + \frac{98n^{11}}{25660800} + 240c_6 \cdot \frac{4}{15}n^{11} + \frac{2|c_7|}{15}n^{11} + c_6 n^7 \end{aligned} \tag{4.49}$$

and one can verify that (4.49) is negative for $n \ge 250$. The remaining cases $8 \le n < 250$ can be checked directly by computer.

Now consider

$$X_{16,5}(it) = e^{-14\pi t} + \sum_{n\ge 8} a_n e^{-2\pi n t} = e^{-14\pi t}\left(1 + \sum_{n \ge 8} a_n e^{-2\pi(n-7)t}\right).$$

Since $a_n < 0$ for $n \ge 8$, the factor in parentheses approaches $-\infty$ as $t \to 0^+$, hence $X_{16,5}$ is not positive. □

Remark 4.5.2. The above proof, especially the negativity of a_n for $n \ge 250$, is verified in Lean 4 with the help of `Claude Opus 4.7`. Since the Ramanujan τ-function is not yet available in Mathlib and Deligne's bound is far from being formalized, we define τ as an abstract function $\mathbb{N} \to \mathbb{Z}$ and take Deligne's bound as an axiom. A proof sketch was provided as input (including the bounds on $\sigma_k(n)$ and the proofs of (4.47) and (4.48)), and the remaining Lean proof was generated by the model; the final steps are mostly handled by the tactics `linarith` and `nlinarith`. The remaining cases are checked with Sage.

Chapter 5

Algebraic proof of modular form inequalities for optimal sphere packings

Using the theory developed in Chapter 3, we give new *algebraic* proofs of the modular form inequalities for optimal sphere packings in dimensions 8 and 24 [80, 19].

5.1 The quasimodular form inequalities and original proofs

Recall that the linear constraints for the magic functions of Viazovska and Cohn et al. reduce to the following inequalities for certain quasimodular forms.

Theorem 5.1.1 (Viazovska [80]). Define[1]

$$\phi_0 = 1728\frac{(E_2E_4 - E_6)^2}{E_4^3 - E_6^2}, \tag{5.1}$$

$$\psi_S = -128\left(\frac{\Theta_3^4 + \Theta_2^4}{\Theta_4^8} + \frac{\Theta_2^4 - \Theta_4^4}{\Theta_3^8}\right). \tag{5.2}$$

Then

$$\phi_0(it) - \frac{36}{\pi^2}\psi_S(it) > 0, \tag{5.3}$$

$$\phi_0(it) + \frac{36}{\pi^2}\psi_S(it) > 0, \tag{5.4}$$

[1]Note that the original inequality is written in terms of $\psi_I(z) = z^2\psi_S(-1/z)$, but we found that the above form is more convenient to work with.

for all $t > 0$.

Theorem 5.1.2 (Cohn et al. [19]). Define

$$\varphi = -\frac{49E_2^2E_4^3 - 25E_2^2E_6^2 - 48E_2E_4^2E_6 - 25E_4^4 + 49E_4E_6^2}{\Delta^2}, \tag{5.5}$$

$$\psi_S = -\frac{\Theta_2^{20}(2\Theta_2^8 + 7\Theta_2^4\Theta_4^4 + 7\Theta_4^8)}{\Delta^2}. \tag{5.6}$$

Then for all $t > 0$,

$$\varphi(it) + \frac{432}{\pi^2}\psi_S(it) < 0, \tag{5.7}$$

$$\varphi(it) - \frac{432}{\pi^2}\psi_S(it) > 0, \tag{5.8}$$

and for all $t \geq 1$,

$$t^{10}\left(\varphi\left(\frac{i}{t}\right) - \frac{432}{\pi^2}\psi_S\left(\frac{i}{t}\right)\right) \geq \frac{725760}{\pi}e^{2\pi t}\left(t - \frac{10}{3\pi}\right). \tag{5.9}$$

Viazovska's proof of Theorem 5.1.1 uses bounds on Fourier coefficients of weakly holomorphic modular forms [8, Proposition 1.12]. For example, the n-th coefficients of ϕ_0 and ψ_S are bounded by

$$|c_{\phi_0}(n)| \leq 2e^{4\pi\sqrt{n}}, \quad n \in \mathbb{Z}_{>0} \tag{5.10}$$

$$|c_{\psi_S}(n)| \leq 2e^{4\pi\sqrt{n}}, \quad n \in \frac{1}{2}\mathbb{Z}_{>0}. \tag{5.11}$$

Using these bounds, one can truncate the series corresponding to (5.3) and (5.4) and bound the error terms to prove the inequalities using interval arithmetic. The above bounds can be obtained using the Hardy–Ramanujan circle method [63] or the nonholomorphic Poincaré series with bounds of Kloosterman sums [62].

The proof of the 24-dimensional case in [19] is slightly different, but still based on numerical analysis. First of all, it is apparent from (5.6) that $\psi_S(it) < 0$ for all $t > 0$, hence it is enough to prove $\varphi(it) < 0$ to prove (5.7). They considered the denominator

$$-\Delta^2\varphi = 25E_4^4 - 49E_6^2E_4 + 48E_6E_4^2E_2 - 49E_4^3E_2^2 + 25E_6^2E_2^2 = \sum_{n\geq 3} a_n q^n \geq 0$$

where $q = e^{-2\pi t}$ with $t > 0$. When $t \geq 1$, $q = e^{-2\pi t} < 1/535$ and the sum of the absolute values of the terms with $n \geq 50$ is at most $10^{-50}q^6$, and we can use Sturm's theorem to conclude that $\sum_{3\leq n<50} a_n q^n + 10^{-50}q^6$ never changes sign in $(0, 1/535)$. Then $\varphi(it) < 0$ follows from the fact that $-\Delta(it)^2\varphi(it) > 0$ for sufficiently large t. For $0 < t < 1$, they used quasimodularity to express $\varphi(i/t)$ in terms of polynomials and other quasimodular functions, and applied a similar technique. Note that this case is more complicated because there are factors of t and π, and they used the first 10 digits of π, i.e. $3.141592653 < \pi < 3.141592654$, to work with exact rational arithmetic. They used PARI/GP for the computation, and Sturm's bound approach avoids the use of hard facts about explicit bounds for Fourier coefficients of weakly holomorphic modular forms such as (5.10) and (5.11). The other two inequalities (5.8) and (5.9) are also proved similarly.

5.2 Romik's proof

In [65], Romik gave a simpler proof of the inequalities in dimension 8, which does not depend on any interval arithmetic but only on comparing certain mathematical constants. Here we briefly summarize the idea.

First, he noticed that (5.3) follows from $\phi_0(it) > 0$ and $\psi_S(it) < 0$ separately, where the first inequality is clear from its definition $\phi_0 = (E_2E_4 - E_6)^2/\Delta$; by the product formula, $\Delta(it) = e^{-2\pi t}\prod_{n\geq 1}(1 - e^{-2\pi n t})^{24} > 0$. The second inequality $\psi_S(it) < 0$ is equivalent to $\psi_I(it) = -t^2\psi_S(i/t) > 0$, where

$$\psi_I = 128\left(\frac{\Theta_3^4 + \Theta_4^4}{\Theta_2^8} + \frac{\Theta_4^4 - \Theta_2^4}{\Theta_3^8}\right). \tag{5.12}$$

Using Jacobi's identity, Romik rewrote ψ_I as

$$\psi_I = \frac{128}{\Theta_3^4}\frac{(1-\lambda)(2+\lambda+2\lambda^2)}{\lambda^2} \tag{5.13}$$

where λ is the modular lambda function (2.35). Now the positivity of $\psi_I(it)$ follows from $0 < \lambda(it) < 1$, and this completes the proof of (5.3).

For (5.4), we treat the cases $t \geq 1$ and $0 < t < 1$ separately, as in [80] and [19]. Also, we need explicit values of the modular forms at $z = i$ (corresponding to $t = 1$), such as

$$E_2(i) = \frac{3}{\pi}, \quad E_4(i) = \frac{3\Gamma(1/4)^6}{64\pi^6}, \quad E_6(i) = 0. \tag{5.14}$$

(See [65, Eq. 16] for the explicit values of theta functions.) For $t \geq 1$, (5.4) is equivalent to $f(it) < g(it)$, where

$$f = \frac{\pi^2}{18}(E_2E_4 - E_6)^2 \tag{5.15}$$
$$g = \Theta_4^8(\Theta_3^{12} + \Theta_4^4\Theta_3^8 + \Theta_2^8\Theta_4^4 - \Theta_2^{12}). \tag{5.16}$$

Both $f(z)$ and $g(z)$ have nonnegative Fourier coefficients starting as

$$f(z) = 28800\pi^2q^2 + 1036800\pi^2q^3 + 14169600\pi^2q^4 + \cdots \tag{5.17}$$
$$g(z) = 20480q^{\frac{3}{2}} + 2015232q^{\frac{5}{2}} + 41656320q^{\frac{7}{2}} + \cdots \tag{5.18}$$

(here $q = e^{2\pi iz}$). In particular, both $t \mapsto e^{3\pi t}f(it)$ and $t \mapsto e^{3\pi t}g(it)$ are monotone decreasing in t, and the desired inequality follows from

$$e^{3\pi t}f(it) \leq e^{3\pi}f(i) = e^{3\pi}\frac{9\Gamma(1/4)^6}{8192\pi^{12}} < 13130.48 < 20480 < e^{3\pi t}g(it). \tag{5.19}$$

The proof for $0 < t < 1$ is much more complicated, and it uses delicate decompositions of modular forms appearing in $\tilde{f}(it) = t^{-2}f(i/t)$ and $\tilde{g}(it) = t^{-2}g(i/t)$, which include polynomial terms in t. Nevertheless, the proof is still based on the monotonicity of certain functions and their explicit values at $t = 1$.

5.3 New proof in dimension 8

Now, we introduce our new proof of Theorem 5.1.1, based on the theory of positive quasimodular forms developed in Chapter 3. First, we define the following (quasi)modular forms:

$$F = (E_2E_4 - E_6)^2, \tag{5.20}$$
$$G = H_2^3(2H_2^2 + 5H_2H_4 + 5H_4^2). \tag{5.21}$$

Using Jacobi's identity (2.25) and (2.34), one can easily check that $F = \Delta\phi_0$ and $G = -2\Delta\psi_S$. In particular, $\psi_S(it) < 0$, which is equivalent to $G(it) > 0$, is now apparent from (5.21) and gives another proof of (5.3).

The second inequality (5.4) is much more interesting, and it is equivalent to

$$F(it) < \frac{18}{\pi^2}G(it) \Leftrightarrow \frac{F(it)}{G(it)} < \frac{18}{\pi^2}. \tag{5.22}$$

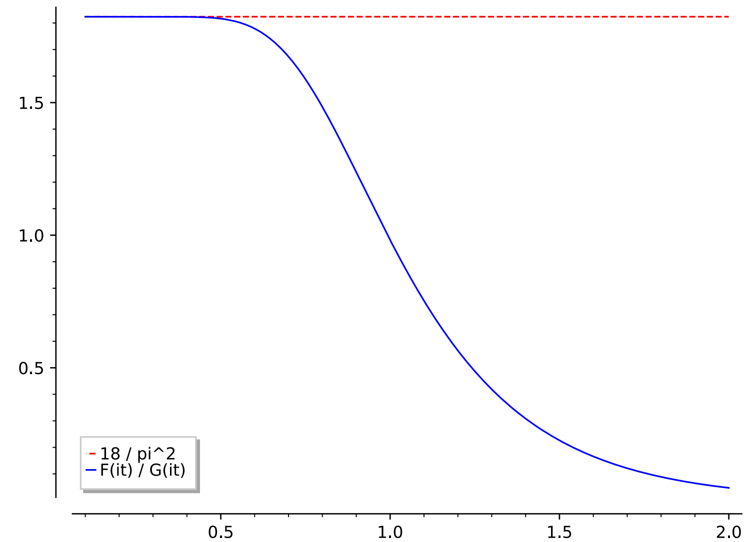


Figure 5.1: Graph of the quotient $F(it)/G(it)$ as a function of $t > 0$ for $d = 8$.

In fact, one can plot the graph of the quotient $t \mapsto \frac{F(it)}{G(it)}$ (e.g., using Sage) and obtain the beautiful plot in Figure 5.1.

This strongly suggests that we should (i) prove monotonicity of the quotient and (ii) compute the limit as $t \longrightarrow 0^+$, and both turned out to be true (Propositions 5.3.1 and 5.3.3).

Proposition 5.3.1. The function $t \mapsto \frac{F(it)}{G(it)}$ is strictly decreasing on $t > 0$.

Proof. It is enough to show that the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{F(it)}{G(it)}\right) = -2\pi\frac{F'(it)G(it) - F(it)G'(it)}{G(it)^2}$$

is negative, which is equivalent to the positivity of (recall that $F(it) > 0$ and $G(it) > 0$)

$$F'G - FG' = (\partial_{10}F)G - F(\partial_{10}G) =: \mathcal{L}_{1,0} \tag{5.23}$$

which has weight 24, level $\Gamma(2)$, and depth 2. Here we give two different proofs of $\mathcal{L}_{1,0} > 0$.

First proof. F and G satisfy the following differential identities:

$$\partial_{10}^2 F = \frac{5}{6}E_4 F + a\Delta X_{4,2}, \tag{5.24}$$

$$\partial_{10}^2 G = \frac{5}{6}E_4 G - b\Delta H_2, \tag{5.25}$$

where $a = 12^3 \cdot 100$ and $b = 640$. Combining these with (2.53), we get

$$\partial_{22}\mathcal{L}_{1,0} = (\partial_{10}^2 F)G - F(\partial_{10}^2 G) = \Delta(aX_{4,2}G + bH_2F) > 0. \tag{5.26}$$

By Corollary 3.3.2, it is enough to show that $\mathcal{L}_{1,0}(it) > 0$ for sufficiently large $t > 0$. This can be done by comparing the vanishing orders at the cusp. From $E_2E_4 - E_6 = 3E_4' = 720q + O(q^2)$, $H_2 = 16q^{\frac{1}{2}} + O(q^{\frac{3}{2}})$, and $H_4 = 1 + O(q^{\frac{1}{2}})$, we have

$$F = 720^2 q^2 + O(q^3), \quad G = 16^3 \cdot 5q^{\frac{3}{2}} + O(q^2)$$

and

$$\frac{F'}{F} = \frac{2 \cdot 720^2 q^2 + O(q^3)}{720^2 q^2 + O(q^3)} = 2 + O(q), \quad \frac{G'}{G} = \frac{\frac{3}{2} \cdot 16^3 \cdot 5q^{\frac{3}{2}} + O(q^2)}{16^3 \cdot 5q^{\frac{3}{2}} + O(q^2)} = \frac{3}{2} + O(q^{\frac{1}{2}}).$$

Since $2 > \frac{3}{2}$, we get $\frac{F'(it)}{F(it)} > \frac{G'(it)}{G(it)}$ for sufficiently large $t > 0$, which is equivalent to $\mathcal{L}_{1,0}(it) > 0$.

Second proof. Using (2.54)-(2.56) and (2.52), one can check that the inequality (5.23) factors as

$$(H_2 + H_4)^2 H_4^2 (E_2E_4 - E_6)\left(E_4 - \frac{1}{2}E_2(H_2 + 2H_4)\right) > 0 \tag{5.27}$$

for $z = it$ with $t > 0$. Since the first three factors are all positive, it is enough to prove that the last factor is positive, i.e.

$$E_4(it) - \frac{1}{2}E_2(it)(H_2(it) + 2H_4(it)) > 0. \tag{5.28}$$

With (2.32), it can be decomposed as

$$E_4 - \frac{1}{2}E_2(H_2 + 2H_4) = \frac{3}{4}H_2^2 + \frac{1}{4}(H_2 + 2H_4)(H_2 + 2H_4 - 2E_2)$$

and the second summand is completely positive, since (from (2.27) and (2.29))

$$H_2 + 2H_4 - 2E_2 = (H_2 + 2H_4 - 2) + 2(1 - E_2) = 2\sum_{n \geq 1} r_4(2n)q^n + 48\sum_{n \geq 1} \sigma_1(n)q^n.$$

□

Remark 5.3.2. One can also use Fourier expansion to check the positivity of $\mathcal{L}_{1,0}(it)$ for sufficiently large $t > 0$; we have

$$\mathcal{L}_{1,0} = 5308416000q^{\frac{7}{2}} + 50960793600q^{\frac{9}{2}} - 528718233600q^{\frac{11}{2}} + O(q^{\frac{13}{2}})$$

and the first nonzero coefficient is positive.

Proposition 5.3.3.

$$\lim_{t\to 0^+} \frac{F(it)}{G(it)} = \frac{18}{\pi^2}. \tag{5.29}$$

Proof. We have

$$\lim_{t\to 0^+} \frac{F(it)}{G(it)} = \lim_{t\to\infty} \frac{F(i/t)}{G(i/t)}.$$

By using the transformation laws of Eisenstein series and the theta functions (2.40), (2.8), (2.9), and (2.24), we get

$$\begin{aligned} F\left(\frac{i}{t}\right) &= t^{12}F(it) - \frac{12t^{11}}{\pi}(E_2(it)E_4(it) - E_6(it))E_4(it) + \frac{36t^{10}}{\pi^2}E_4(it)^2, \\ G\left(\frac{i}{t}\right) &= t^{10}H_4(it)^3(2H_4(it)^2 + 5H_4(it)H_2(it) + 5H_2(it)^2). \end{aligned}$$

Since F, $E_2E_4 - E_6$ and H_2 are cusp forms, we have $\lim_{t\to\infty} t^k A(it) = 0$ when $A(z)$ is one of these forms and $k \geq 0$. From $\lim_{t\to\infty} E_4(it) = 1 = \lim_{t\to\infty} H_4(it)$, we get

$$\begin{aligned} \lim_{t\to\infty} \frac{F(i/t)}{G(i/t)} &= \lim_{t\to\infty} \frac{t^{12}F(it) - \frac{12t^{11}}{\pi}(E_2(it)E_4(it) - E_6(it))E_4(it) + \frac{36t^{10}}{\pi^2}E_4(it)^2}{t^{10}H_4(it)^3(2H_4(it)^2 + 5H_4(it)H_2(it) + 5H_2(it)^2)} \\ &= \lim_{t\to\infty} \frac{t^2F(it) - \frac{12t}{\pi}(E_2(it)E_4(it) - E_6(it))E_4(it) + \frac{36}{\pi^2}E_4(it)^2}{H_4(it)^3(2H_4(it)^2 + 5H_4(it)H_2(it) + 5H_2(it)^2)} \\ &= \frac{18}{\pi^2}. \end{aligned}$$

□

5.4 New proof in dimension 24

It is natural to ask whether the above strategy also works for proving Theorem 5.1.2, and the answer is yes. In particular, the proof of (5.8) follows the same idea as the proof of (5.4).

We will abuse notation and write $F = -\Delta^2\varphi$ and $G = -\Delta^2\psi_S$ for the numerators of (5.5) and (5.6), i.e.

$$F = 49E_2^2E_4^3 - 25E_2^2E_6^2 - 48E_2E_4^2E_6 - 25E_4^4 + 49E_4E_6^2, \tag{5.30}$$

$$G = H_2^5(H_2^2 + 7H_2H_4 + 7H_4^2). \tag{5.31}$$

Then the inequalities (5.7), (5.8) and (5.9) are equivalent to

$$F(it) + \frac{432}{\pi^2} G(it) > 0, \tag{5.32}$$

$$F(it) - \frac{432}{\pi^2} G(it) < 0, \tag{5.33}$$

$$t^{10}\left(-\frac{F(i/t)}{\Delta(i/t)^2} + \frac{432}{\pi^2}\frac{G(i/t)}{\Delta(i/t)^2}\right) \geq \frac{725760}{\pi} e^{2\pi t}\left(t - \frac{10}{3\pi}\right). \tag{5.34}$$

***Easy* inequality (5.32)**

That $G(it) > 0$ is immediate from its definition (5.31), as noted in [19, p. 1028]. Hence, it is enough to prove $F(it) > 0$, which is not clear a priori. However, this follows from the following surprising identity, combined with Corollary 3.3.2.

Lemma 5.4.1. We have

$$\partial_{14} F = 6706022400 \cdot X_{6,1} X_{12,1} \in \mathcal{QM}_{18}^{2,++}. \tag{5.35}$$

Proof. The identity follows from direct calculations. (Complete) positivity follows from Theorem 4.2.1. □

Corollary 5.4.2. $F(it) > 0$ for all $t > 0$.

Proof. This follows from Lemma 5.4.1 and Corollary 3.3.2. Note that F has a Fourier expansion

$$F = 3657830400q^3 + 138997555200q^4 + 2567796940800q^5 + O(q^6)$$

and its first nonzero Fourier coefficient is positive. □

Remark 5.4.3. In fact, F is a constant multiple of f_{16} that appears in the family of Feigenbaum–Grabner–Hardin [26, Proposition 5.1]. The authors already proved that the functions f_w are *completely* positive for $w \leq 94$ [26, Remark 6.3], and they conjectured that all the forms in the family are completely positive [26, Conjecture 1]. However, their proof uses approximations based on Jenkins and Rouse's explicit bound on Fourier coefficients [38].

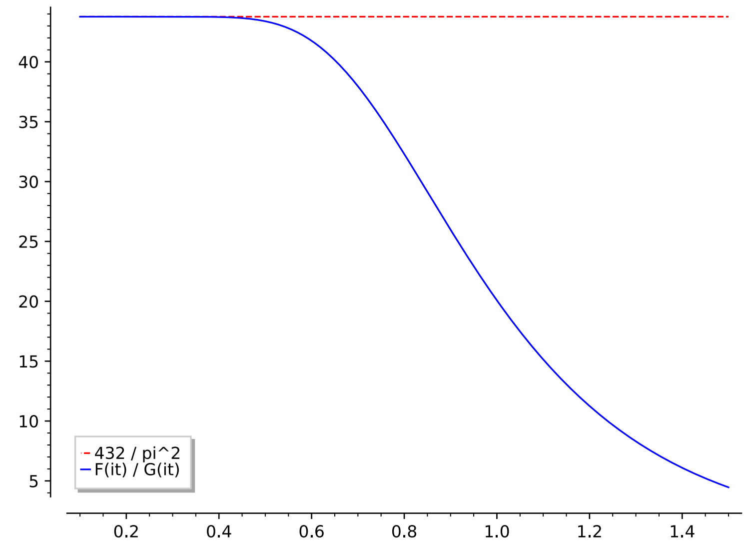


Figure 5.2: Graph of the quotient $F(it)/G(it)$ as a function of $t > 0$ for $d = 24$.

***Hard* inequality** (5.33)

As noted above, the strategy in Section 5.3 works perfectly for (5.8), too. Figure 5.2 shows that the function $t \mapsto F(it)/G(it)$ follows the same behavior as the quotient in Figure 5.1, so we only need to show that (i) the quotient is monotone decreasing and (ii) the limit as $t \to 0^+$ equals the desired constant.

Proposition 5.4.4. The function $t \mapsto \frac{F(it)}{G(it)}$ is strictly decreasing on $t > 0$.

Proof. The idea is similar to the second proof of Proposition 5.3.1. It is enough to show that

$$F'G - FG' = (\partial_{14}F)G - F(\partial_{14}G) =: \mathcal{L}_{1,0}$$

is positive; it has weight 32, level $\Gamma(2)$, and depth 2. F and G satisfy the following differential equations, similar to (5.24) and (5.25):

$$\partial_{14}^2 F = \frac{14}{9} E_4 F + c\Delta X_{8,2} \tag{5.36}$$

$$\partial_{14}^2 G = \frac{14}{9} E_4 G \tag{5.37}$$

where $c = 5486745600$. Taking the Serre derivative ∂_{30}, by (2.53), (5.36), and (5.37), we have

$$\partial_{30}\mathcal{L}_{1,0} = \mathcal{L}_{2,0} := (\partial_{14}^2 F)G - F(\partial_{14}^2 G) = c\Delta X_{8,2} G > 0 \tag{5.38}$$

As in the proof of Proposition 5.3.1, to prove $\mathcal{L}_{1,0}(it) > 0$ for sufficiently large t, it is enough to show that the vanishing order of F at infinity is larger than that of G. The vanishing order of F is 3 and that of G is $\frac{5}{2}$, hence $\mathcal{L}_{1,0}(it)$ is positive for sufficiently large $t > 0$. Hence Corollary 3.3.2 applies and we get the positivity of $\mathcal{L}_{1,0}$. □

Remark 5.4.5. $\mathcal{L}_{1,0}$ has a Fourier expansion

$$\mathcal{L}_{1,0} = 13424296093286400q^{\frac{11}{2}} + 494781198866841600q^{\frac{13}{2}} + O(q^{\frac{15}{2}})$$

where the first nonzero Fourier coefficient is positive, so $\mathcal{L}_{1,0}(it) > 0$ for sufficiently large $t > 0$.

Proposition 5.4.6.

$$\lim_{t\to 0^+} \frac{F(it)}{G(it)} = \frac{432}{\pi^2}. \tag{5.39}$$

Proof. The same argument as in Proposition 5.3.3 applies. We omit the details. □

Remark 5.4.7. In Section 6.4, we give an alternative proof of the monotonicity (i.e. the positivity of $\mathcal{L}_{1,0}$) which is similar to the first proof of Proposition 5.3.1.

Remark 5.4.8. As in the $d = 8$ case, (5.36) and (5.37) describe the action of the operator $L_{2,14}$ of type $(14, 18)$ in the sense of [43, 56] on F and G. In particular, G is a solution of the linear differential equation $L_{2,14}G = 0$.

Harder **inequality (5.34)**

The last inequality (5.34) is more involved than the previous inequalities because of the presence of the non-modular terms t^{10} and $e^{2\pi t}$. We first replace the exponential term $e^{2\pi t}$ with $1/\Delta$ using the following inequality.

Lemma 5.4.9. For $t > 0$, we have

$$\Delta(it) < e^{-2\pi t}. \tag{5.40}$$

Proof. This directly follows from the product formula of Δ,

$$\Delta(it) = e^{-2\pi t} \prod_{n\geq 1}(1 - e^{-2\pi n t})^{24} < e^{-2\pi t}.$$

□

The inequality (5.34) is true for $1 \leq t \leq \frac{10}{3\pi}$, since the left-hand side (resp. the right-hand side) is nonnegative (resp. nonpositive) on this range (by (5.33)). Hence it is enough to show this for $t > \frac{10}{3\pi}$. On this range, we can bound the right-hand side of (5.34) using (5.40)

$$\frac{725760}{\pi} e^{2\pi t}\left(t - \frac{10}{3\pi}\right) < \frac{725760}{\pi}\frac{1}{\Delta(it)}\left(t - \frac{10}{3\pi}\right) = \frac{725760}{\pi}\frac{t^{12}}{\Delta(i/t)}\left(t - \frac{10}{3\pi}\right)$$

and the inequality reduces to

$$\begin{aligned} t^{10}\left(-\frac{F(i/t)}{\Delta(i/t)^2} + \frac{432}{\pi^2}\frac{G(i/t)}{\Delta(i/t)^2}\right) &> \frac{725760}{\pi}\frac{t^{12}}{\Delta(i/t)}\left(t - \frac{10}{3\pi}\right) \\ \Leftrightarrow \frac{1}{t^2}\left(-\frac{F(i/t)}{\Delta(i/t)} + \frac{432}{\pi^2}\frac{G(i/t)}{\Delta(i/t)}\right) &> \frac{725760}{\pi}\left(t - \frac{10}{3\pi}\right) \end{aligned}$$

for $t > \frac{10}{3\pi}$. If we replace t by $1/t$, the last inequality becomes

$$\begin{aligned} t^3\left(-\frac{F(it)}{\Delta(it)} + \frac{432}{\pi^2}\frac{G(it)}{\Delta(it)}\right) &> \frac{725760}{\pi}\left(1 - \frac{10t}{3\pi}\right) \\ \Leftrightarrow \frac{432}{\pi^2} - \frac{F(it)}{G(it)} &> \frac{725760\Delta(it)}{G(it)}\left(\frac{1}{\pi t^3} - \frac{10}{3\pi^2 t^2}\right) \end{aligned} \tag{5.41}$$

for $0 < t < \frac{3\pi}{10}$. From Proposition 5.4.4, we know that the left-hand side of (5.41) is monotone increasing in t. Our main observation is that the difference between the two sides of (5.41) is also monotone increasing; see Figure 5.3.

Proposition 5.4.10. The function

$$g(t) := \frac{432}{\pi^2} - \frac{F(it)}{G(it)} - \frac{725760\Delta(it)}{G(it)}\left(\frac{1}{\pi t^3} - \frac{10}{3\pi^2 t^2}\right)$$

is monotone increasing in t for $0 < t < \frac{3\pi}{10}$ and $\lim_{t\to 0^+} g(t) = 0$. In particular, we have $g(t) > 0$ for all $0 < t < \frac{3\pi}{10}$.

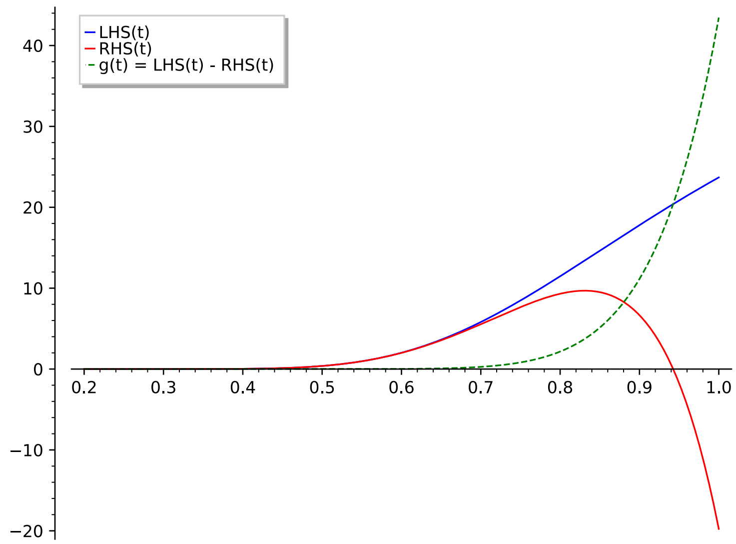


Figure 5.3: Graph of LHS(t), RHS(t), and $g(t) = \mathrm{LHS}(t) - \mathrm{RHS}(t)$ of (5.41).

Proof. After writing the limit as $\lim_{t\to 0^+} g(t) = \lim_{t\to\infty} g(1/t)$, we can compute the limit from Proposition 5.4.6 and the fact that $\Delta|_{12}S = \Delta$ is a cusp form but $G|_{14}S = -H_4^5(7H_2^2 + 7H_2H_4 + 2H_4^2)$ is not, so the third term in $g(t)$ vanishes as $t \to 0^+$. We omit the details and focus on the monotonicity part. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{F(it)}{G(it)}\right) = -2\pi\frac{\mathcal{L}_{1,0}(it)}{G(it)^2}$$

and by $\Delta' = E_2\Delta$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\Delta(it)}{G(it)}\left(\frac{1}{\pi t^3} - \frac{10}{3\pi^2 t^2}\right)\right] \\
&= (-2\pi)\frac{\Delta(it)(E_2(it)G(it) - G'(it))}{G(it)^2}\left(\frac{1}{\pi t^3} - \frac{10}{3\pi^2 t^2}\right) + \frac{\Delta(it)}{G(it)}\left(-\frac{3}{\pi t^4} + \frac{20}{3\pi^2 t^3}\right) \\
&= (2\pi)\frac{\Delta(it)}{G(it)^2}\left[(\partial_{12}G)(it)\left(\frac{1}{\pi t^3} - \frac{10}{3\pi^2 t^2}\right) - G(it)\left(\frac{3}{2\pi^2 t^4} - \frac{10}{3\pi^3 t^3}\right)\right],
\end{aligned}$$

so $\mathrm{d}g/\mathrm{d}t > 0$ if and only if (after factoring out $1/G^2$)

$$\widetilde{\mathcal{L}}_{1,0}(it) := \mathcal{L}_{1,0}(it) - 725760\Delta(it)\left[(\partial_{12}G)(it)\left(\frac{1}{\pi t^3} - \frac{10}{3\pi^2 t^2}\right) - G(it)\left(\frac{3}{2\pi^2 t^4} - \frac{10}{3\pi^3 t^3}\right)\right] > 0.$$

We have $\widetilde{\mathcal{L}}_{1,0}(\frac{3\pi i}{10}) > 0$, since Proposition 5.4.4 gives $\mathcal{L}_{1,0}(\frac{3\pi i}{10}) > 0$ and when $t = \frac{3\pi}{10}$,

$$(\partial_{12}G)(it)\left(\frac{1}{\pi t^3} - \frac{10}{3\pi^2 t^2}\right) - G(it)\left(\frac{3}{2\pi^2 t^4} - \frac{10}{3\pi^3 t^3}\right) = -G\left(\frac{3\pi i}{10}\right)\cdot\left(\frac{5000}{81\pi^6}\right) < 0.$$

From Proposition 3.3.1 (see also Remark 3.3.3), it is enough to show that its Serre derivative

$$\partial_{30}\widetilde{\mathcal{L}}_{1,0}(it) = \widetilde{\mathcal{L}}_{1,0}'(it) - \frac{5}{2}E_2(it)\widetilde{\mathcal{L}}_{1,0}(it) = -\frac{1}{2\pi}\frac{\mathrm{d}\widetilde{\mathcal{L}}_{1,0}(it)}{\mathrm{d}t} - \frac{5}{2}E_2(it)\widetilde{\mathcal{L}}_{1,0}(it)$$

is positive (i.e. $t \mapsto \widetilde{\mathcal{L}}_{1,0}(it)/\eta(it)^{60}$ is a monotone decreasing function in t) on $0 < t < \frac{3\pi}{10}$. Recall $\partial_{30}\mathcal{L}_{1,0} = c\Delta X_{8,2}G$ (5.38). Using (2.53), $\partial_{12}\Delta = 0$, and (5.37), one can check that the Serre derivative of the second term of $\widetilde{\mathcal{L}}_{1,0}$ is 725760Δ times

$$\begin{aligned}&\partial_{18}\left[(\partial_{12}G)(it)\left(\frac{1}{\pi t^3} - \frac{10}{3\pi^2 t^2}\right) - G(it)\left(\frac{3}{2\pi^2 t^4} - \frac{10}{3\pi^3 t^3}\right)\right]\\ &= \left[\frac{37E_4(it) - E_2(it)^2}{24}\left(\frac{1}{\pi t^3} - \frac{10}{3\pi^2 t^2}\right) + E_2(it)\left(\frac{3}{4\pi^2 t^4} - \frac{5}{3\pi^3 t^3}\right) - \left(\frac{3}{\pi^3 t^5} - \frac{5}{\pi^4 t^4}\right)\right]G(it),\end{aligned} \tag{5.42}$$

so ΔG factors out from $\partial_{30}\widetilde{\mathcal{L}}_{1,0}(it) > 0$ and it reduces to the positivity of

$$7560X_{8,2}(it) - \frac{37E_4(it) - E_2(it)^2}{24}\left(\frac{1}{\pi t^3} - \frac{10}{3\pi^2 t^2}\right) - E_2(it)\left(\frac{3}{4\pi^2 t^4} - \frac{5}{3\pi^3 t^3}\right) + \left(\frac{3}{\pi^3 t^5} - \frac{5}{\pi^4 t^4}\right). \tag{5.43}$$

Let $h(t)$ be the above function in (5.43). We have (see Table C.1)

$$7560X_{8,2} = \frac{-7E_2^2E_4 + 2E_2E_6 + 5E_4^2}{48}$$

and similarly

$$\begin{aligned}7560X_{8,2}|_S &= 7560\left(X_{8,2} + \frac{7E_2E_4 - E_6}{30240\pi t} - \frac{E_4}{1440\pi^2 t^2}\right)\\ &= \frac{-7E_2^2E_4 + 2E_2E_6 + 5E_4^2}{48} + \frac{7E_2E_4 - E_6}{4\pi t} - \frac{21E_4}{4\pi^2 t^2}.\end{aligned}$$

Then

$$\begin{aligned}&t^{-8}h\left(\frac{1}{t}\right)\\ &= 7560\left(X_{8,2}(it) + \frac{7E_2(it)E_4(it) - E_6(it)}{30240\pi t} - \frac{E_4(it)}{1440\pi^2 t^2}\right)\\ &\quad - \frac{1}{24}\left(37E_4(it) - E_2(it)^2 + \frac{12E_2(it)}{\pi t} - \frac{36}{\pi^2 t^2}\right)\left(\frac{1}{\pi t} - \frac{10}{3\pi^2 t^2}\right)\\ &\quad - \left(-E_2(it) + \frac{6}{\pi t}\right)\left(\frac{3}{4\pi^2 t^2} - \frac{5}{3\pi^3 t^3}\right) + \left(\frac{3}{\pi^3 t^3} - \frac{5}{\pi^4 t^4}\right)\end{aligned}$$

$$= 7560 X_{8,2}(it) + \frac{1}{\pi t}\left(\frac{7E_2(it)E_4(it) - E_6(it)}{4} - \frac{37E_4(it) - E_2(it)^2}{24}\right) + \frac{1}{\pi^2 t^2}\left(-\frac{4E_4(it) + 5E_2(it)^2}{36} + \frac{E_2(it)}{4}\right) \tag{5.44}$$

and since $X_{8,2}$ is (completely) positive (Proposition 4.4.2), it is enough to show that (5.44) is positive, i.e. show the following inhomogeneous inequality (after factoring out $1/\pi t$)

$$\frac{7E_2(it)E_4(it) - E_6(it)}{4} - \frac{37E_4(it) - E_2(it)^2}{24} - \frac{1}{\pi t}\left(\frac{5E_2(it)^2 + 4E_4(it)}{36} - \frac{1}{4}E_2(it)\right) > 0 \tag{5.45}$$

for $t \geq \frac{10}{3\pi}$. We can further reduce it to an inequality with only quasimodular terms (i.e. no rational terms) with the following lemma.

Lemma 5.4.11. The following (nonhomogeneous) quasimodular forms are completely positive:

$$J_1 = \frac{5}{36}E_2^2 + \frac{1}{9}E_4 - \frac{1}{4}E_2, \tag{5.46}$$

$$J_2 = E_2 - E_6. \tag{5.47}$$

Proof. By using the Fourier expansions of E_2, E_4, E_6, and (2.42), we can compute the Fourier expansions of the above forms explicitly as

$$J_1 = \frac{5}{3}E_2' - \frac{1}{4}E_2 + \frac{1}{4}E_4 = \sum_{n\geq 1}(60\sigma_3(n) - 40n\sigma_1(n) + 6\sigma_1(n))q^n,$$

$$J_2 = \sum_{n\geq 1}(504\sigma_5(n) - 24\sigma_1(n))q^n.$$

The complete positivity of J_1 follows from the trivial estimates $\sigma_3(n) \geq n^3$ and $\sigma_1(n) \leq 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \leq n^2$, and that of J_2 follows from $504\sigma_5(n) - 24\sigma_1(n) = \sum_{d|n}(504d^5 - 24d) > 0$. □

By Lemma 5.4.11, for $t \geq \frac{10}{3\pi}$ we have

$$\begin{aligned} &\frac{7E_2(it)E_4(it) - E_6(it)}{4} - \frac{37E_4(it) - E_2(it)^2}{24} - \frac{1}{\pi t}\left(\frac{5E_2(it)^2 + 4E_4(it)}{36} - \frac{E_2(it)}{4}\right) \\ &\geq \frac{7E_2(it)E_4(it) - E_6(it)}{4} - \frac{37E_4(it) - E_2(it)^2}{24} - \frac{3}{10}\left(\frac{5E_2(it)^2 + 4E_4(it)}{36} - \frac{E_2(it)}{4}\right) \\ &= \frac{7E_2(it)E_4(it) - E_6(it)}{4} - \frac{63}{40}E_4(it) + \frac{3}{40}E_2(it) \end{aligned} \tag{5.48}$$

$$> \frac{7E_2(it)E_4(it) - E_6(it)}{4} - \frac{63}{40}E_4(it) + \frac{3}{40}E_6(it) \tag{5.49}$$

$$= \frac{7}{4}\left(E_2(it)E_4(it) - \frac{1}{10}E_6(it) - \frac{9}{10}E_4(it)\right) =: \frac{7}{4}J_3 \tag{5.50}$$

where the positivity results for J_1 and J_2 are used in (5.48) and (5.49), respectively. Now, we can prove the positivity of (5.50) (i.e. J_3) as follows. As in Lemma 5.4.11, we can compute the Fourier expansion of J_3 as

$$\begin{aligned} J_3 &= E_2E_4 - \frac{1}{10}E_6 - \frac{9}{10}E_4 \\ &= 3E_4' + \frac{9}{10}E_6 - \frac{9}{10}E_4 \\ &= \sum_{n\geq 1}\left(720n\sigma_3(n) - \frac{2268}{5}\sigma_5(n) - 216\sigma_3(n)\right)q^n \\ &=: \sum_{n\geq 1} a_n q^n. \end{aligned}$$

We have $a_1 = \frac{252}{5} > 0$. For $n \geq 2$,

$$n\sigma_3(n) \leq n(1^3 + 2^3 + \cdots + n^3) = \frac{n^3(n+1)^2}{4} \leq \frac{9}{16}n^5 < \frac{9}{16}\sigma_5(n) < \frac{2268}{720\cdot 5}\sigma_5(n)$$

and we get $a_n < 0$. From this observation, the function

$$t \mapsto e^{2\pi t}J_3(it) = a_1 + \sum_{n\geq 2} a_n e^{-2\pi(n-1)t}$$

is monotone increasing, and by (5.14) we have

$$e^{2\pi t}J_3(it) \geq e^{2\pi}J_3(i) = e^{2\pi}\left(\frac{3}{\pi} - \frac{9}{10}\right)E_4(i) > 0 \Rightarrow J_3(it) > 0$$

for $t \geq 1$, hence for $t > \frac{10}{3\pi}$. □

Remark 5.4.12. The estimate $\pi < \frac{10}{3}$ is used in the proof (e.g., (5.48)), and this can be verified *geometrically* (without calculators) by considering the area of a regular octagon circumscribed about a unit circle:

$$\pi < 8\tan\left(\frac{\pi}{8}\right) = 8(\sqrt{2} - 1) < \frac{10}{3}.$$

Remark 5.4.13. The inequalities (5.41) and (5.43) are "homogeneous" if one regards $\frac{1}{t} = \frac{i}{z}$ and $\frac{1}{\pi}$ as "weight 1" objects, which makes sense if we consider the transformation law of E_2 (2.40). However, we had to apply the substitution $t \leftrightarrow \frac{1}{t}$ and prove the nonhomogeneous inequalities (5.45)-(5.50) instead. It would be interesting if one could prove the inequality (5.34) in a purely homogeneous way.

5.5 Formalization of the proof

The algebraic nature of the proof gives it a natural advantage for formalization. There is an ongoing project on formalizing Viazovska's proof in Lean 4 [55], led by Sidharth Hariharan [72]. In particular, adapting the new proof strategy in Section 5.3 made formalization much easier: we can completely avoid interval arithmetic, nontrivial bounds on Fourier coefficients such as (5.10) or (5.11) (which would require the Hardy–Ramanujan circle method to prove), and comparisons of complicated mathematical constants. Details can be found in Appendix B, which also includes various simplifications of the proofs in [80] to facilitate formalization.

Chapter 6

Inequalities involving polynomials and quasimodular forms

In this chapter, we study the monotonicity of functions of the form

$$t \mapsto t^m F(it) \tag{6.1}$$

for certain (completely) positive quasimodular forms F. As an application, we construct infinitely many positive quasimodular forms of higher level (Section 6.4) and give an alternative proof of Proposition 5.4.4. Finally, we give a simple algebraic proof of inequality (3b) of [20, p. 1067], which is used to prove the universal optimality of the Leech lattice (Section 6.6).

6.1 Small-weight examples

For certain low-weight quasimodular forms $F(z)$, it is possible to prove monotonicity of $t^m F(it)$ via *Lambert series*, i.e. expansions of the form

$$F(z) = \sum_{n \geq 1} b_n \frac{q^n}{(1-q^n)^k}.$$

Once we know that $b_n \geq 0$ for all n, monotonicity of $t^m F(it)$ reduces to that of

$$t \mapsto t^m \frac{e^{-t}}{(1-e^{-t})^k},$$

which is often easy to check. For example, we can prove monotonicity of the following functions.

Lemma 6.1.1. The function $t \mapsto t^2(1 - E_2(it))$ is monotone decreasing for $t > 0$.

Proof. From (2.39), we have

$$1 - E_2(z) = 24 \sum_{n \geq 1} \sigma_1(n) q^n = 24 \sum_{m,d \geq 1} d q^{dm} = 24 \sum_{m \geq 1} \frac{q^m}{(1 - q^m)^2},$$

so it is enough to check that the following function is monotone in t:

$$g_2(t) := t^2 \frac{e^{-t}}{(1 - e^{-t})^2}.$$

This follows from

$$\frac{\mathrm{d}}{\mathrm{d}t} g_2(t) = -\frac{e^t t (e^t (t-2) + (t+2))}{(e^t - 1)^3}$$

and

$$e^t(t-2) + (t+2) > 0 \Leftrightarrow \tanh\left(\frac{t}{2}\right) < \frac{t}{2},$$

where the last inequality follows by applying $h(u) = u - \tanh(u)$ to $u = t/2$, since $h'(u) = \tanh^2(u) > 0$ and $h(0) = 0$. □

Lemma 6.1.2. The function $t \mapsto t^3 X_{4,2}(it)$ is monotone decreasing for $t > 0$.

Proof. We can write $X_{4,2}$ as

$$X_{4,2}(z) = \sum_{n \geq 1} n \sigma_1(n) q^n = \sum_{m \geq 1} \sum_{d \geq 1} d^2 m q^{dm} = \sum_{m \geq 1} \frac{m q^m (1 + q^m)}{(1 - q^m)^3},$$

where the last equality follows from $\sum_{k \geq 1} k^2 x^k = \frac{x(1+x)}{(1-x)^3}$. Hence the monotonicity of $t^3 X_{4,2}(it)$ reduces to that of

$$g_3(t) := t^3 \frac{e^{-t}(1 + e^{-t})}{(1 - e^{-t})^3}.$$

The derivative of $g_3(t)$ is

$$\frac{\mathrm{d}g_3}{\mathrm{d}t} = -\frac{e^t t^2 (t(e^{2t} + 4e^t + 1) - 3(e^{2t} - 1))}{(e^t - 1)^4} = -\frac{3t^2 e^t (e^{2t} + 4e^t + 1)}{(e^t - 1)^4} \left(\frac{t}{3} - \frac{e^{2t} - 1}{e^{2t} + 4e^t + 1} \right).$$

If we put

$$h_3(t) := \frac{t}{3} - \frac{e^{2t} - 1}{e^{2t} + 4e^t + 1},$$

then $\mathrm{d}g_3/\mathrm{d}t < 0$ if and only if $h_3(t) > 0$, which follows from $h_3(0) = 0$ and

$$\frac{\mathrm{d}h_3}{\mathrm{d}t} = \frac{(e^t - 1)^4}{3(e^{2t} + 4e^t + 1)^2} > 0.$$

□

Lemma 6.1.3. The function $t \mapsto t^2(E_2(2it) - E_2(it))$ is monotone decreasing for $t > 0$.

Proof. We can write $E_2(2z) - E_2(z)$ as

$$E_2(2z) - E_2(z) = 24\sum_{n\geq 1}\sigma_1(n)q^n - 24\sum_{n\geq 1}\sigma_1(n)q^{2n} = 24\sum_{n\geq 1}\left(\frac{q^n}{(1-q^n)^2} - \frac{q^{2n}}{(1-q^{2n})^2}\right),$$

so it is enough to check that the following function is monotone in t:

$$g_{2,2}(t) := t^2\left(\frac{e^{-t}}{(1-e^{-t})^2} - \frac{e^{-2t}}{(1-e^{-2t})^2}\right) = t^2\frac{e^{-t}(1+e^{-t}+e^{-2t})}{(1-e^{-2t})^2}.$$

The derivative of $g_{2,2}(t)$ is

$$\frac{\mathrm{d}g_{2,2}}{\mathrm{d}t} = -\frac{2te^t(e^{4t}+2e^{3t}+6e^{2t}+2e^t+1)}{(e^{2t}-1)^3}\left[\frac{t}{2} - \frac{(e^t+1)(e^{3t}-1)}{(e^{4t}+2e^{3t}+6e^{2t}+2e^t+1)}\right].$$

If we put

$$h_{2,2}(t) := \frac{t}{2} - \frac{(e^t+1)(e^{3t}-1)}{(e^{4t}+2e^{3t}+6e^{2t}+2e^t+1)},$$

then $\mathrm{d}g_{2,2}/\mathrm{d}t < 0$ if and only if $h_{2,2}(t) > 0$, which follows from $h_{2,2}(0) = 0$ and

$$\frac{\mathrm{d}h_{2,2}}{\mathrm{d}t} = \frac{(e^t-1)^4(e^t+1)^4(e^{2t}+4e^t+1)}{2(e^{4t}+2e^{3t}+6e^{2t}+2e^t+1)^2} > 0.$$

□

Lemma 6.1.4. The function $t \mapsto t^6\mathrm{X}_{8,1}(it)$ is monotone decreasing for $t > 0$.

Proof. We can write $X_{8,1}$ as

$$\begin{aligned} X_{8,1}(z) &= \sum_{n\geq 1} n\sigma_5(n)q^n = \sum_{m\geq 1}\sum_{d\geq 1} d^6 m q^{dm} \\ &= \sum_{m\geq 1}\frac{mq^m(q^{5m}+57q^{4m}+302q^{3m}+302q^{2m}+57q^m+1)}{(1-q^m)^7} \end{aligned}$$

so it is enough to check that the following function is monotone in t:

$$g_6(t) := t^6 \frac{e^{-t}(e^{-5t} + 57e^{-4t} + 302e^{-3t} + 302e^{-2t} + 57e^{-t} + 1)}{(1 - e^{-t})^7}.$$

This follows from

$$\frac{\mathrm{d}g_6}{\mathrm{d}t} = -\frac{e^t t^5 (tP(e^t) - Q(e^t))}{(e^t - 1)^8}$$

where $P(x)$ and $Q(x)$ are polynomials

$$\begin{aligned} P(x) &= x^6 + 120x^5 + 1191x^4 + 2416x^3 + 1191x^2 + 120x + 1, \\ Q(x) &= 6(x^6 + 56x^5 + 245x^4 - 245x^2 - 56x - 1). \end{aligned}$$

The inequality $tP(e^t) - Q(e^t) > 0$ for $t > 0$ follows from

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[t - \frac{Q(e^t)}{P(e^t)}\right] = \frac{R(e^t)}{P(e^t)^2} > 0,$$

where

$$\begin{aligned} R(x) = {} & (x^6 - 72x^5)^2 + (1 - 72x)^2 + 246(x^{10} + x^2) + 23408(x^9 + x^3) \\ & + 342687(x^8 + x^4) + 754464(x^7 + x^5) + 1377108x^6 \end{aligned}$$

which is positive for all $x > 1$, and $Q(1) = 0$. □

Lemma 6.1.5. The function $t \mapsto t^8 X_{10,1}(it)$ is monotone decreasing for $t > 0$.

Proof. We can write $X_{10,1}$ as

$$\begin{aligned} X_{10,1}(z) &= \sum_{n \geq 1} n\sigma_7(n) q^n = \sum_{m \geq 1} \sum_{d \geq 1} d^8 m q^{dm} \\ &= \sum_{m \geq 1} \frac{mq^m((q^{7m} + 1) + 247(q^{6m} + q^m) + 4293(q^{5m} + q^{2m}) + 15619(q^{4m} + q^{3m}))}{(1 - q^m)^9} \end{aligned}$$

so it is enough to check that the following function is monotone in t:

$$g_8(t) := t^8 \frac{e^{-t}((e^{-7t} + 1) + 247(e^{-6t} + e^{-t}) + 4293(e^{-5t} + e^{-2t}) + 15619(e^{-4t} + e^{-3t}))}{(1 - e^{-t})^9}.$$

Its derivative is of the form

$$-\frac{e^{-9t} t^7 (tP(e^t) - Q(e^t))}{(1 - e^{-t})^{10}}$$

where $P(x)$ and $Q(x)$ are polynomials

$$\begin{aligned}P(x) &= (x^8+1)+502(x^7+x)+14608(x^6+x^2)+88234(x^5+x^3)+156190x^4,\\ Q(x) &= 8(x^8-1+246(x^7-x)+4046(x^6-x^2)+11326(x^5-x^3)),\end{aligned}$$

and it is enough to show that

$$f(t) := tP(e^t) - Q(e^t) > 0 \text{ for } t > 0. \tag{6.2}$$

To prove this, we consider its Taylor expansion at $t = 0$:

$$f(t) = \sum_{k=0}^{8}(a_k t + b_k)e^{kt} = \sum_{k=0}^{8}\left(\sum_{n\geq 1}\frac{(na_k + kb_k)k^{n-1}}{n!}t^n + b_k\right) = \sum_{n\geq 0}\frac{c_n}{n!}t^n$$

where $P(x) = \sum_{0\leq k\leq 8} a_k x^k$, $Q(x) = \sum_{0\leq k\leq 8} b_k x^k$, $c_0 = f(0) = 8 > 0$ and

$$\begin{aligned}c_n &= \sum_{k=1}^{8}(na_k + kb_k)k^{n-1}\\ &= (n-64)8^{n-1} + (502n-13376)7^{n-1} + (14608n-194208)6^{n-1}\\ &\quad + (88234n-453040)5^{n-1} + 156190n4^{n-1} + (88234n+271824)3^{n-1}\\ &\quad + (14608n+64736)2^{n-1} + (502n+1968).\end{aligned}$$

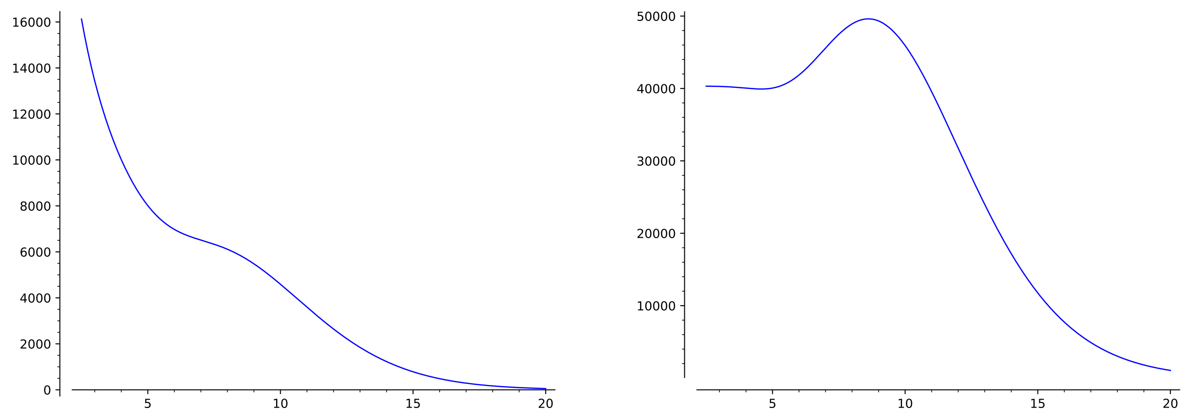
Thus it is enough to show that $c_n > 0$ for all $n \geq 1$. For $n \geq 65$, all the linear coefficients are positive, so $c_n > 0$. We can check $c_n > 0$ for $1 \leq n \leq 64$ directly. □

Remark 6.1.6. Figure 6.1 shows the graphs of $g_8(t)$ and $g_9(t)$ on $2.5 \leq t \leq 20$, where $g_9(t)$ is the function obtained by replacing t^8 in $g_8(t)$ with t^9. The function $g_9(t)$ is not monotone decreasing, which can be checked rigorously by comparing the limit of $g_9(t)$ as $t \to 0^+$ and the value $g_9(10)$. In fact, the function $t \mapsto t^9 X_{10,1}(it)$ is not monotone decreasing; see Remark 6.2.9.

Remark 6.1.7. We also note that the above proof of (6.2) was suggested by `ChatGPT-5.2 Pro` and `AxiomProver`, where the latter model provided a formal proof in Lean 4.

Unfortunately, such an approach does not always work. For example, the function

$$t \mapsto t^4(E_4(it) - 1)$$



(a) $g_8(t)$ (b) $g_9(t)$

Figure 6.1: Graphs of $g_8(t)$ and $g_9(t)$ on $2.5 \leq t \leq 20$.

is monotone decreasing for $t > 0$ (see Corollary 6.2.6). However, the following Lambert series expansion

$$E_4(z) - 1 = 240 \sum_{n \geq 1} \sigma_3(n) q^n = 240 \sum_{m,d \geq 1} d^2 q^{dm} = 240 \sum_{m \geq 1} \frac{q^m(1 + 4q^m + q^{2m})}{(1 - q^m)^4}$$

is not enough to prove monotonicity, since the function

$$t \mapsto t^4 \frac{e^{-t}(1 + 4e^{-t} + e^{-2t})}{(1 - e^{-t})^4}$$

is not monotone decreasing; it is $6 + \frac{t^4}{120} + O(t^5)$ near $t = 0$. Similarly, the function

$$t \mapsto t^5 X_{6,1}(it)$$

is monotone decreasing for $t > 0$ (Corollary 6.2.5). However, the Lambert series expansion of $X_{6,1}$ is

$$X_{6,1}(z) = \sum_{m \geq 1} \sum_{d \geq 1} d^4 m q^{dm} = \sum_{m \geq 1} \frac{m q^m (1 + 11q^m + 11q^{2m} + q^{3m})}{(1 - q^m)^5}$$

but the function

$$t \mapsto t^5 \frac{e^{-t}(1 + 11e^{-t} + 11e^{-2t} + e^{-3t})}{(1 - e^{-t})^5}$$

is not monotone decreasing; it is $24 + \frac{t^6}{252} + O(t^7)$ near $t = 0$.

Another non-example is the weight 12 and depth 1 extremal quasimodular form

$$X_{12,1} = -\frac{E_{10}'}{277200} - \frac{\Delta}{1050} = \frac{1}{1050}\sum_{n\geq 1}(n\sigma_9(n) - \tau(n))q^n$$

which does not admit a nice Lambert series expansion, because of the presence of the term $\tau(n)$. Monotonicity of the relevant functions will be proved in the following section with a different approach (Corollary 6.2.6 and Corollary 6.2.7).

6.2 Two sufficient conditions and more examples

We introduce two simple sufficient conditions (Proposition 6.2.1 and 6.2.10) which are useful for proving monotonicity of $t^m F(it)$.

Proposition 6.2.1. Let F be a quasimodular form and let $m \in \mathbb{R}_{>0}$. Assume that

1. F and F' are positive quasimodular forms,
2. $$\lim_{t\to 0^+} \frac{F(it)}{tF'(it)} = \frac{2\pi}{m},$$
3. $(m+1)(F')^2 - mF''F$ is positive.

Then the function $t \mapsto t^m F(it)$ is monotone decreasing on $(0,\infty)$. Moreover, m is the largest possible constant such that $t^m F(it)$ is monotone decreasing.

Proof. Consider the function

$$h(t) := \frac{F(it)}{F'(it)}.$$

The limit condition implies that $h(0) := \lim_{t\to 0^+} h(t) = 0$ and $h'(0) = \frac{2\pi}{m}$. In other words, the tangent line of $h(t)$ at $t=0$ is $l(t) = \frac{2\pi}{m}t$. Direct computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(t^m F(it)) = mt^{m-1}F(it) + t^m(-2\pi)F'(it) = mt^{m-1}F'(it)\left(h(t) - l(t)\right),$$

so it is enough to show that $h(t) < l(t)$ for all $t > 0$. Consider their difference $g(t) = l(t) - h(t)$. The limit condition implies that $g(0) = 0$ and it is enough to show that $g(t)$ is monotone increasing, and this is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}g(t) = \frac{2\pi}{m} + 2\pi \cdot \frac{(F'(it))^2 - F''(it)F(it)}{(F'(it))^2} > 0 \Leftrightarrow (m+1)(F')^2 - mF''F > 0.$$

The last claim follows from considering the behavior of $h(t)$ near $t = 0$. □

Remark 6.2.2. The modular form $(m+1)(F')^2 - mF''F$ is a constant multiple of the Rankin–Cohen bracket [64, 10] of F with itself. More precisely, Martin and Royer [54] generalized the notion of Rankin–Cohen brackets for quasimodular forms as

$$\Phi_{n;k,s;\ell,t}(F,G) = \sum_{r=0}^{n} (-1)^r \binom{k-s+n-1}{n-r}\binom{\ell-t+n-1}{r}(D^rF)(D^{n-r}G)$$

for $0 \le s \le k/2$ and $0 \le t \le \ell/2$. This matches the usual Rankin–Cohen bracket $[F,G]_n^{(k,\ell)}$ when $s = t = 0$. One can directly check that

$$\begin{aligned}\Phi_{2;m,0;m,0}(F,F) &= [F,F]_2^{(m,m)} \\ &= \sum_{r=0}^{2} (-1)^r \binom{m+1}{2-r}\binom{m+1}{r}(D^rF)(D^{2-r}F) \\ &= (m+1)mF''F - (m+1)^2(F')^2\end{aligned}$$

and the third condition of Proposition 6.2.1 is equivalent to the negativity of $\Phi_{2;m,0;m,0}(F,F)$. In particular, when F has weight w and depth s, then for $m = w - s$

$$(m+1)(F')^2 - mF''F = -\frac{1}{m+1}\Phi_{2;w,s;w,s}(F,F).$$

This is a quasimodular form of weight $2w + 4$ and depth at most $2s$.

Remark 6.2.3. For any completely positive quasimodular form F, we have $F''F - (F')^2 > 0$. In fact, this holds for any completely monotone function (see, e.g., [28]).

As corollaries, we can prove monotonicity of several functions involving extremal quasimodular forms, such as the following (which cannot be proved by the Lambert series approach in Section 6.1).

Proposition 6.2.4. Let $X_{w,1}$ be the normalized extremal quasimodular form of weight w and depth 1. Then

$$\lim_{t \to 0^+} \frac{X_{w,1}(it)}{tX'_{w,1}(it)} = \frac{2\pi}{w-1} \tag{6.3}$$

$$\lim_{t\to 0^+} t^{w-1} X_{w,1}(it) = -\frac{6(-1)^{\frac{w}{2}}\beta_{w-2}}{\pi} \tag{6.4}$$

for all $w \geq 6$.

Proof. The transformation law gives

$$X_{w,1}\left(-\frac{1}{z}\right) = z^w A_w(z) + \left(z^2 E_2(z) - \frac{6iz}{\pi}\right) z^{w-2} B_{w-2}(z) = z^w X_{w,1}(z) - \frac{6iz^{w-1}}{\pi} B_{w-2}(z)$$

and $z = it$ gives

$$X_{w,1}\left(\frac{i}{t}\right) = (-1)^{\frac{w}{2}} \left[t^w X_{w,1}(it) - \frac{6t^{w-1}}{\pi} B_{w-2}(it)\right]. \tag{6.5}$$

By differentiating $X_{w,1}$ and rewriting in terms of Serre derivatives, we have

$$\begin{aligned}
X'_{w,1} &= A'_w + E'_2 B_{w-2} + E_2 B'_{w-2} \\
&= \partial_w A_w + \frac{w}{12} E_2 A_w + \frac{E_2^2 - E_4}{12} B_{w-2} + E_2 \left(\partial_{w-2} B_{w-2} + \frac{w-2}{12} E_2 B_{w-2}\right) \\
&= \left(\partial_w A_w - \frac{1}{12} E_4 B_{w-2}\right) + E_2 \left(\frac{w}{12} A_w + \partial_{w-2} B_{w-2}\right) + E_2^2 \cdot \frac{w-1}{12} B_{w-2} \\
&=: \widetilde{A}_{w+2} + E_2 \widetilde{B}_w + E_2^2 \widetilde{C}_{w-2},
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{A}_{w+2} &:= \partial_w A_w - \frac{1}{12} E_4 B_{w-2} \in \mathcal{M}_{w+2}(\mathrm{SL}_2(\mathbb{Z})), \\
\widetilde{B}_w &:= \frac{w}{12} A_w + \partial_{w-2} B_{w-2} \in \mathcal{M}_w(\mathrm{SL}_2(\mathbb{Z})), \\
\widetilde{C}_{w-2} &:= \frac{w-1}{12} B_{w-2} \in \mathcal{M}_{w-2}(\mathrm{SL}_2(\mathbb{Z})).
\end{aligned}$$

Then applying the transformation law with $z = it$ gives

$$\begin{aligned}
&X'_{w,1}\left(\frac{i}{t}\right) \\
&= (-1)^{\frac{w}{2}+1} \left[t^{w+2} X'_{w,1}(it) + \frac{t^{w+1}}{\pi}(-6\widetilde{B}_w(it) - (w-1) B_{w-2}(it) E_2(it)) + \frac{t^w}{\pi^2} \cdot 3(w-1) B_{w-2}(it)\right] \\
&= (-1)^{\frac{w}{2}+1} \left[t^{w+2} X'_{w,1}(it) + \frac{t^{w+1}}{\pi} \left(-\frac{w}{2} X_{w,1}(it) - 6B'_w(it)\right) + \frac{t^w}{\pi^2} \cdot 3(w-1) B_{w-2}(it)\right] \qquad (6.6)
\end{aligned}$$

By combining (6.5) and (6.6), we have

$$
\begin{aligned}
\lim_{t\to 0^+} \frac{X_{w,1}(it)}{tX'_{w,1}(it)} &= \lim_{t\to\infty} \frac{X_{w,1}(i/t)}{\frac{1}{t}X'_{w,1}(i/t)} \\
&= \lim_{t\to\infty} -\frac{t^w X_{w,1}(it) - \frac{6t^{w-1}}{\pi} B_{w-2}(it)}{t^{w+1}X'_{w,1}(it) + \frac{t^w}{\pi}\left(-\frac{w}{2}X_{w,1}(it) - 6B'_{w-2}(it)\right) + \frac{t^{w-1}}{\pi^2}\cdot 3(w-1)B_{w-2}(it)} \\
&= -\lim_{t\to\infty} \frac{tX_{w,1}(it) - \frac{6}{\pi}B_{w-2}(it)}{t^2X'_{w,1}(it) + \frac{t}{\pi}\left(-\frac{w}{2}X_{w,1}(it) - 6B'_{w-2}(it)\right) + \frac{1}{\pi^2}\cdot 3(w-1)B_{w-2}(it)}.
\end{aligned}
$$

Since $X_{w,1}, X'_{w,1}$ and $-\frac{w}{2}X_{w,1} - 6B'_{w-2}$ are cusp forms, the limits of the terms involving them vanish as $t \to \infty$. By Lemma 4.3.2, B_{w-2} is not a cusp form and the final limit equals

$$
-\frac{-\frac{6}{\pi}\cdot \beta_{w-2}}{\frac{1}{\pi^2}\cdot 3(w-1)\cdot \beta_{w-2}} = \frac{2\pi}{w-1}
$$

which proves (6.3). The limit (6.4) follows from (6.5) and the cuspidality of $X_{w,1}$. □

Corollary 6.2.5. $t \mapsto t^5 X_{6,1}(it)$ is monotone decreasing for $t > 0$.

Proof. We know that $X_{6,1}$ is completely positive. The limit condition follows from Proposition 6.2.4 with $w = 6$. Direct computation shows that

$$
6(X'_{6,1})^2 - 5X''_{6,1}X_{6,1} = \Delta X_{4,2}
$$

which is positive. □

Corollary 6.2.6. $t \mapsto t^4(E_4(it) - 1)$ is monotone decreasing for $t > 0$.

Proof. The monotonicity is equivalent to

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left[t^4(E_4(it)-1)\right] = 4t^3(E_4(it)-1) - 2\pi t^4 E'_4(it) < 0 \Leftrightarrow E_4(it) - 1 - \frac{\pi t}{6}(E_2(it)E_4(it) - E_6(it)) < 0.
$$

Substituting t with $1/t$ and using the transformation laws, this is equivalent to

$$
\begin{aligned}
&E_4\left(\frac{i}{t}\right) - 1 - \frac{\pi}{6t}\left[E_2\left(\frac{i}{t}\right)E_4\left(\frac{i}{t}\right) - E_6\left(\frac{i}{t}\right)\right] \\
&= t^4E_4(it) - 1 - \frac{\pi}{6t}\left[\left(-t^2E_2(it) + \frac{6t}{\pi}\right)\cdot t^4E_4(it) + t^6E_6(it)\right]
\end{aligned}
$$

$$= -1 + \frac{\pi t^5}{6}(E_2(it)E_4(it) - E_6(it))$$
$$= -1 + 120\pi t^5 X_{6,1}(it) < 0.$$

The last inequality follows from Corollary 6.2.5 and $\lim_{t\to 0^+} t^5 X_{6,1}(it) = \frac{1}{120\pi}$ by Proposition 6.2.4. □

Corollary 6.2.7. $t \mapsto t^{11}X_{12,1}(it)$ is monotone decreasing for $t > 0$.

Proof. We know that $X_{12,1}$ is completely positive. The limit condition follows from Proposition 6.2.4 with $w = 12$. Direct computation shows that

$$12(X'_{12,1})^2 - 11X''_{12,1}X_{12,1} = \frac{1}{2^{10}\cdot 3^6\cdot 5^2\cdot 7^2}\cdot \Delta F,$$

where

$$F = 49E_2^2E_4^3 - 25E_2^2E_6^2 - 48E_2E_4^2E_6 - 25E_4^4 + 49E_4E_6^2$$

is the quasimodular form of weight 16 and depth 2 that appears in the Leech lattice packing (5.30), which is positive by Corollary 5.4.2. □

Corollary 6.2.8. $t \mapsto t^{13}X_{14,1}(it)$ is monotone decreasing for $t > 0$.

Proof. Direct computation shows that

$$14(X'_{14,1})^2 - 13X''_{14,1}X_{14,1} = 4\Delta^2 X_{8,2},$$

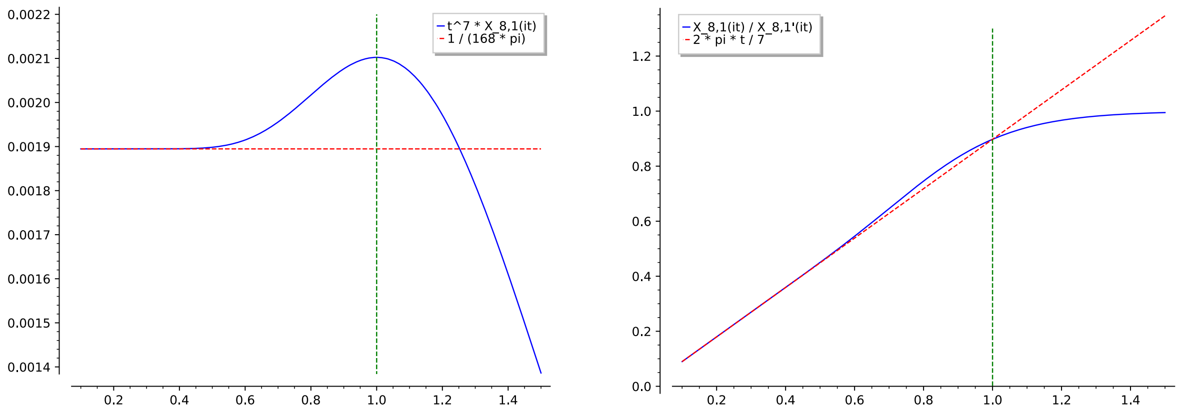
where $X_{8,2}$ is the normalized extremal quasimodular form of weight 8 and depth 2, which is completely positive by Proposition 4.4.2. □

Remark 6.2.9. Note that $t \mapsto t^7X_{8,1}(it)$ and $t \mapsto t^9X_{10,1}(it)$ are not monotone decreasing (see Figures 6.2 and 6.3). The former function has a local maximum at $t = 1$; we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(t^7X_{8,1}(it)) = t^6(7X_{8,1}(it) - 2\pi t X'_{8,1}(it))$$

and from $E_2(i) = \frac{3}{\pi}$ and $E_6(i) = 0$,

$$7X_{8,1}(i) - 2\pi X'_{8,1}(i) = 7\cdot\frac{E_4(i)^2 - E_2(i)E_6(i)}{1008} - 2\pi\left(\frac{-E_2(i)^2E_6(i) + 2E_2(i)E_4(i)^2 - E_4(i)E_6(i)}{1728}\right)$$



(a) $t^7 X_{8,1}(it)$ (b) $X_{8,1}(it)/X'_{8,1}(it)$

Figure 6.2: Graphs of $t^7 X_{8,1}(it)$ and $X_{8,1}(it)/X'_{8,1}(it)$ with asymptotes as $t \longrightarrow 0^+$.

$$= E_4(i)^2 \left(\frac{7}{1008} - \frac{2\pi \cdot 2 \cdot \frac{3}{\pi}}{1728} \right) = 0.$$

For the latter function, note that

$$X_{10,1} = \frac{E_4(E_2E_4 - E_6)}{720}, \qquad X'_{10,1} = \frac{9E_2^2E_4^2 - 18E_2E_4E_6 + 5E_4^3 + 4E_6^2}{8640}.$$

By Proposition 6.2.4, the limit as $t \longrightarrow 0^+$ is $-\frac{6(-1)^{\frac{10}{2}}\beta_8}{\pi} = \frac{1}{120\pi}$, while the value at $t = 1$ is

$$X_{10,1}(i) = \frac{E_2(i)E_4(i)^2}{720} = \frac{1}{720} \cdot \frac{3}{\pi} \cdot \frac{9\Gamma(1/4)^{16}}{4096\pi^{12}} > \frac{1}{120\pi}.$$

The second proposition shows that monotonicity of $t^{m+1}F'(it)$ implies that of $t^m F(it)$.

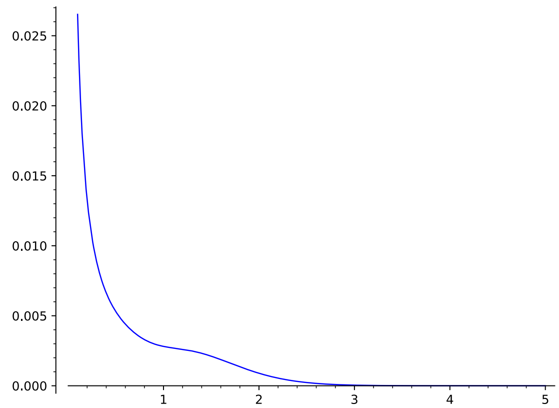
Proposition 6.2.10. Let F be a quasimodular cusp form. If $t \mapsto t^{m+1}F'(it)$ is monotone decreasing, then $t \mapsto t^m F(it)$ is also monotone decreasing.

Proof. For $a > 1$ and $t > 0$, define

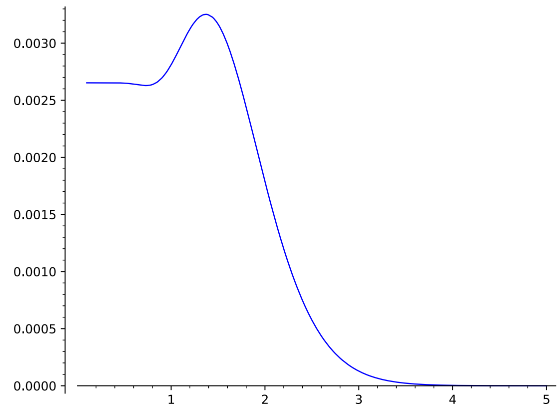
$$g_a(t) := F(it) - a^m F(iat).$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} g_a(t) = (-2\pi)(F'(it) - a^{m+1}F'(iat)),$$



(a) $t^8 X_{10,1}(it)$



(b) $t^9 X_{10,1}(it)$

Figure 6.3: Graphs of $t^8 X_{10,1}(it)$ and $t^9 X_{10,1}(it)$ on $0.1 \le t \le 5$.

hence $g_a(t)$ is monotone decreasing for all $a > 1$. By cuspidality, we have $\lim_{t\to\infty} g_a(t) = 0$ and $g_a(t) \ge 0$, hence $t \mapsto t^m F(it)$ is monotone decreasing. □

Corollary 6.2.11. The following functions are monotone decreasing:

$$\begin{aligned} t &\mapsto t^7 X_{8,2}(it), \\ t &\mapsto t^9 X_{10,2}(it), \\ t &\mapsto t^{11} X_{12,2}(it), \\ t &\mapsto t^{13} X_{14,2}(it). \end{aligned}$$

Proof. By the identity $X'_{8,2} = 2X_{4,2}X_{6,1}$ (4.38), Lemma 6.1.2, and Corollary 6.2.5, the function $t^8 X'_{8,2}(it) = 2 \cdot t^3 X_{4,2}(it) \cdot t^5 X_{6,1}(it)$ is monotone decreasing. Proposition 6.2.10 then gives that $t^7 X_{8,2}(it)$ is also monotone decreasing. The other cases follow from the same argument with the identities (4.39), (4.40) and (4.41) of Proposition 4.4.2, Lemma 6.1.2, Lemma 6.1.4, Corollary 6.2.5, Corollary 6.2.7, and Proposition 6.2.10. □

6.3 Monotonicity of $t^{w-1} X_{w,1}(it)$

Based on Corollaries 6.2.5, 6.2.7, and 6.2.8, and on numerical experiments, we conjecture the following monotonicity property of extremal quasimodular forms of depth 1.

Conjecture 6.3.1. For all even $w \geq 6$ except for $w = 8, 10$, the function

$$t \mapsto t^{w-1} X_{w,1}(it)$$

is monotone decreasing for $t > 0$.

The exceptional cases $w = 8$ and $w = 10$ are discussed in Remark 6.2.9. Although we cannot prove this conjecture in general, we can prove the following weaker claims. First, the function is monotone decreasing near $t = 0$.

Proposition 6.3.2. For $w \geq 12$, the function $t \mapsto t^{w-1} X_{w,1}(it)$ is monotone decreasing near $t = 0$.

Proof. By Corollaries 6.2.7 and 6.2.8, we may assume that $w \geq 16$. It is enough to show that the derivative is negative near $t = 0$, which is equivalent to

$$2\pi t X'_{w,1}(it) - (w-1) X_{w,1}(it) > 0$$

near $t = 0$. By substituting t with $1/t$ and using (6.5) and (6.6), this is equivalent to

$$\begin{aligned}
&\frac{2\pi}{t} X'_{w,1}\left(\frac{i}{t}\right) - (w-1) X_{w,1}\left(\frac{i}{t}\right) \\
&= (-1)^{\frac{w}{2}} \Bigg[-2\pi t^{w+1} X'_{w,1}(it) + t^w \left(w X_{w,1}(it) + 12 B'_{w-2}(it) \right) - \frac{2t^{w-1}}{\pi} \cdot 3(w-1) B_{w-2}(it) \\
&\qquad - (w-1) t^w X_{w,1}(it) + \frac{6(w-1)t^{w-1}}{\pi} B_{w-2}(it) \Bigg] \\
&= (-1)^{\frac{w}{2}} t^w \left(-2\pi t X'_{w,1}(it) + X_w(it) + 12 B'_{w-2}(it) \right) > 0
\end{aligned}$$

for sufficiently large t. Since $X_{w,1}(it)$ and $X'_{w,1}(it)$ are both $O(e^{-4\pi t})$, the positivity for large enough t is equivalent to $\lim_{t\to\infty} (-1)^{\frac{w}{2}} B'_{w-2}(it) = (-1)^{\frac{w}{2}} \beta_{w-2,1} > 0$, which follows from Lemma 4.3.2 and 4.3.3. ∎

Also, we can prove monotonicity of $t^{a_w} X_{w,1}(it)$ for the smaller exponent

$$a_w = w - \lceil w/6 \rceil \leq w - 1$$

using the recurrence relations in Theorem 4.2.1 (Theorem 6.3.3).

Theorem 6.3.3. For all even $w \geq 6$, the function

$$t \mapsto t^{w-\lceil w/6 \rceil} X_{w,1}(it)$$

is monotone decreasing for $t > 0$.

Proof. Let $a_w := w - \lceil w/6 \rceil$. By induction, one can easily check that a_w satisfies the following recurrence relations for all $w \geq 12$ with $w \equiv 0 \pmod 6$:

$$a_w = \min\{a_{w-4} + 4, a_{w-6} + 5\} \tag{6.7}$$

$$a_{w+2} = \min\{a_{w-2} + 4, a_{w-4} + 5\} \tag{6.8}$$

$$a_{w+4} = \min\{a_w + 4, a_{w-2} + 5, a_{w-4} + 7\} \tag{6.9}$$

For example, when $w = 6k$, we have $a_w = 5k$, $a_{w-4} + 4 = 6k - 4 - k + 4 = 5k$, and $a_{w-6} + 5 = 5(k-1) + 5 = 5k$, hence (6.7) holds. The other cases can be checked similarly. Then the theorem follows for $w = 6, 8, 10$ by Corollary 6.2.5, Lemma 6.1.4, and Lemma 6.1.5. Assume that it holds for $w-6, w-4, w-2$ for some $w \geq 12$ with $w \equiv 0 \pmod 6$. By the induction hypothesis and Lemma 6.1.4, both of the functions

$$t^5 X_{6,1}(it) \cdot t^{a_{w-4}} X_{w-4,1}(it), \quad t^6 X_{8,1}(it) \cdot t^{a_{w-6}} X_{w-6,1}(it)$$

are monotone decreasing, and by (4.7) and (6.7), the function

$$t^{a_w+1} X'_{w,1}(it) := \frac{5w}{72} \cdot t^5 X_{6,1}(it) \cdot t^{a_w-4} X_{w-4,1}(it) + \frac{7w}{72} \cdot t^6 X_{8,1}(it) \cdot t^{a_w-5} X_{w-6,1}(it)$$

is also monotone decreasing. Therefore, by Proposition 6.2.10, the function $t^{a_w} X_{w,1}(it)$ is also monotone decreasing. We can similarly prove the result for $t^{a_{w+2}} X_{w+2,1}(it)$ and $t^{a_{w+4}} X_{w+4,1}(it)$ using (4.8), (4.9), (6.8), and (6.9). □

6.4 Application: Positive quasimodular forms of higher levels

Let F be a positive quasimodular form of level 1. Then for $N \in \mathbb{Z}_{\geq 1}$, $F^{\langle N \rangle}(z) := F(Nz)$ is also a positive quasimodular form of level $\Gamma_0(N)$ (Proposition 3.4.1). If we further assume that F' is positive, i.e. $t \mapsto F(it)$ is monotone decreasing for $t > 0$, then $F(z) - F^{\langle N \rangle}(z)$ is also positive (Proposition 3.4.2). The following corollary shows that one can obtain a stronger result if the monotonicity of $t \mapsto t^m F(it)$ is known for some $m > 0$.

Corollary 6.4.1. Let F be a quasimodular form and let $m \in \mathbb{R}_{>0}$. Assume that $t \mapsto t^m F(it)$ is monotone decreasing for $t > 0$. Then for $N \in \mathbb{Z}_{\geq 1}$, $F(z) - N^m F^{\langle N \rangle}(z)$ is a positive quasimodular form of level $\Gamma_0(N)$. Also, if $\lim_{t\to 0^+} t^m F(it)$ exists and is positive, then the constant N^m is optimal, i.e. for any $c > N^m$, $F(z) - cF^{\langle N \rangle}(z)$ is not positive.

Proof. Since $t^m F(it)$ is monotone decreasing,

$$t^m F(it) - (Nt)^m F(iNt) = t^m (F(it) - N^m F(iNt))$$

is positive for $t > 0$. If $\lim_{t\to 0^+} t^m F(it) = L > 0$, then

$$\lim_{t\to 0^+} \frac{F(it)}{F(iNt)} = \lim_{t\to 0^+} \frac{N^m \cdot t^m F(it)}{(Nt)^m F(iNt)} = \frac{N^m \cdot L}{L} = N^m$$

which shows the optimality of the constant. □

Example 6.4.2. From Lemmas 6.1.2 and 6.1.3 and Corollaries 6.2.5 and 6.2.7, we obtain the following positive quasimodular forms by taking $N = 2$ in Corollary 6.4.1:

$$\begin{aligned}
P_1(z) &:= X_{4,2}(z) - 8X_{4,2}(2z) \\
&= \sum_{n\geq 1} n\left(\sigma_1(n) - 4\sigma_1\left(\frac{n}{2}\right)\right) q^n \in \mathcal{QM}_4^{2,+}(\Gamma_0(2)) \\
P_2(z) &:= \frac{-E_2(z) + 5E_2(2z) - 4E_2(4z)}{24} \\
&= \sum_{n\geq 1} \left(\sigma_1(n) - 5\sigma_1\left(\frac{n}{2}\right) + 4\sigma_1\left(\frac{n}{4}\right)\right) q^n \in \mathcal{QM}_2^{1,+}(\Gamma_0(4)) \\
P_3(z) &:= X_{6,1}(z) - 32X_{6,1}(2z) \\
&= \sum_{n\geq 1} n\left(\sigma_3(n) - 16\sigma_3\left(\frac{n}{2}\right)\right) q^n \in \mathcal{QM}_6^{1,+}(\Gamma_0(2)) \\
P_4(z) &:= X_{12,1}(z) - 2^{11}X_{12,1}(2z) \\
&= \frac{1}{1050}\sum_{n\geq 1} \left[n\left(\sigma_9(n) - 2^{10}\sigma_9\left(\frac{n}{2}\right)\right) - \left(\tau(n) - 2^{11}\tau\left(\frac{n}{2}\right)\right)\right] q^n \in \mathcal{QM}_{12}^{1,+}(\Gamma_0(2))
\end{aligned}$$

Note that none of these quasimodular forms is completely positive. For example, the n-th coefficient of $P_1(z)$ for odd n is $n\sigma_1(n)$, which is positive. For even $n = 2^k \cdot m$ with $k \geq 1$ and m odd, the coefficient is

$$n\left(\sigma_1(n) - 4\sigma_1\left(\frac{n}{2}\right)\right) = n \cdot (\sigma_1(2^k)\sigma_1(m) - 4\sigma_1(2^{k-1})\sigma_1(m)) = n \cdot (-2^{k+1} + 3)\sigma_1(m) < 0.$$

In particular, taking the negation of the form with substitution $z \mapsto z+\frac{1}{2}$ gives a completely positive quasimodular form of level $\Gamma_0(4)$:

$$-P_1\left(z+\frac{1}{2}\right) = -X_{4,2}\left(z+\frac{1}{2}\right)+8X_{4,2}\left(2z\right) = \sum_{n\geq 1} n\left(\sigma_1(n)-4\sigma_1\left(\frac{n}{2}\right)\right)(-1)^{n+1}q^n \in \mathcal{QM}_4^{2,++}(\Gamma_0(4)).$$

Similarly, the n-th coefficient of $P_3(z)$ is positive if and only if n is odd, and we get a completely positive quasimodular form of level $\Gamma_0(4)$:

$$-P_3\left(z+\frac{1}{2}\right) = -X_{6,1}\left(z+\frac{1}{2}\right)+32X_{6,1}\left(2z\right) = \sum_{n\geq 1} n\left(\sigma_3(n)-16\sigma_3\left(\frac{n}{2}\right)\right)(-1)^{n+1}q^n \in \mathcal{QM}_6^{1,++}(\Gamma_0(4)).$$

For $P_2(z)$, one can check that the n-th coefficient is positive for odd n, and negative for $n = 2m$ for odd m. When $n = 2^k \cdot m$ with $k \geq 2$, the n-th coefficient is

$$(\sigma_1(2^k)-5\sigma_1(2^{k-1})+4\sigma_1(2^{k-2}))\sigma_1(m) = ((2^{k+1}-1)-5(2^k-1)+4(2^{k-1}-1))\sigma_1(m) = 3\cdot 2^{k-1}\sigma_1(m) > 0.$$

Thus the n-th coefficient of $P_2(z)$ is positive if and only if $n \not\equiv 2 \pmod 4$. If we write $P_2(z) = \sum_{n\geq 1} a_n q^n$, then

$$P_2\left(\frac{z}{2}+\frac{1}{4}\right) = ia_1q^{1/2} - a_2q - ia_3q^{3/2} + a_4q^2 + ia_5q^{5/2} - a_6q^3 - ia_7q^{7/2} + a_8q^4 + \cdots$$
$$P_2\left(\frac{z}{2}-\frac{1}{4}\right) = -ia_1q^{1/2} - a_2q + ia_3q^{3/2} + a_4q^2 - ia_5q^{5/2} - a_6q^3 + ia_7q^{7/2} + a_8q^4 + \cdots$$

This shows that the following quasimodular forms are completely positive:

$$\frac{1}{2}\left(P_2(z) - P_2\left(z+\frac{1}{2}\right)\right) = \frac{-E_2(z)+E_2\left(z+\frac{1}{2}\right)}{48} = \sum_{k\geq 0}\sigma_1(2k+1)q^{2k+1}$$
$$\frac{1}{14}\left(P_2\left(\frac{z}{2}+\frac{1}{4}\right)+P_2\left(\frac{z}{2}-\frac{1}{4}\right)\right) = \frac{1}{24}(6E_2(4z)-5E_2(2z)-E_2(z))$$
$$= \frac{1}{2}\sum_{n\equiv 0\,(\mathrm{mod} 4)}\left(\sigma_1(n)-5\sigma_1\left(\frac{n}{2}\right)+4\sigma_1\left(\frac{n}{4}\right)\right)q^{n/2} + \frac{1}{2}\sum_{n\equiv 2\,(\mathrm{mod} 4)}\left(5\sigma_1\left(\frac{n}{2}\right)-\sigma_1(n)\right)q^{n/2}$$

where this simplification uses (2.57). Both forms are of level $\Gamma_0(4)$.

In the case of $P_4(z)$, we can use Deligne's bound for $\tau(n)$ to study signs of the coefficients. The n-th coefficient of $P_4(z)$ for odd n is positive, since

$$n\sigma_9(n) - \tau(n) > n\sigma_9(n) - \sigma_0(n)n^{\frac{11}{2}} \geq n^{10} - n^{\frac{13}{2}} > 0,$$

by the trivial estimates $\sigma_9(n) > n^9$, $\sigma_0(n) \leq n$, and Deligne's bound $|\tau(n)| < \sigma_0(n)n^{\frac{11}{2}}$. If $n = 2^k \cdot m$ with $k \geq 1$ and m odd, then we can estimate the n-th coefficient using Deligne's bound again:

$$\begin{aligned}
&n\left(\sigma_9(2^k) - 2^{10}\sigma_9(2^{k-1})\right)\sigma_9(m) - \left(\tau(2^k) - 2^{11}\tau(2^{k-1})\right)\tau(m) \\
&= n \cdot \frac{-2^{9(k+1)} + 1023}{511} \cdot \sigma_9(m) - (\tau(2^{k+1}) + 25\tau(2^k))\tau(m) \\
&< 2^k \cdot \frac{-2^{9(k+1)} + 1023}{511} \cdot m\sigma_9(m) + ((k+2)2^{\frac{11(k+1)}{2}} + 25(k+1)2^{\frac{11k}{2}})m^{\frac{13}{2}}
\end{aligned}$$

and the last expression is negative for $k \geq 2$, since

$$2^k \cdot \frac{2^{9(k+1)} - 1023}{511} > 2^{\frac{11k}{2}}((2^{\frac{11}{2}} + 25)k + (2^{\frac{13}{2}} + 25)), \quad m\sigma_9(m) > m^{10} \geq m^{\frac{13}{2}}$$

holds for $k \geq 2$ and all odd $m \geq 1$. Note that the recurrence relation $\tau(2^{k+1}) = \tau(2)\tau(2^k) - 2^{11}\tau(2^{k-1}) = -24\tau(2^k) - 2048\tau(2^{k-1})$ is also used in the estimates above. When $k = 1$ and $n = 2m$, the n-th coefficient is

$$n(\sigma_9(2) - 2^{10}\sigma_9(1)) - (\tau(2) - 2^{11}\tau(1))\tau(m) = -1022m\sigma_9(m) + 2072\tau(m)$$

which is positive for $m = 1$, and negative for $m \geq 2$ by Deligne's bound.

Example 6.4.3. By Theorem 6.3.3, for each even $w \geq 6$ and $N \geq 2$,

$$X_{w,1}(z) - N^{w - \lceil w/6 \rceil} X_{w,1}(Nz)$$

is a positive quasimodular form of level $\Gamma_0(N)$. If Conjecture 6.3.1 holds, then we can take the larger exponent $w - 1$ instead of $w - \lceil w/6 \rceil$.

Example 6.4.4. From Corollary 6.2.11, we obtain the following positive quasimodular forms:

$$\begin{aligned}
\widetilde{X}_{8,2}(z) &:= X_{8,2}(z) - 2^7 X_{8,2}(2z) \in \mathcal{QM}_8^{2,+}(\Gamma_0(2)) \\
\widetilde{X}_{10,2}(z) &:= X_{10,2}(z) - 2^9 X_{10,2}(2z) \in \mathcal{QM}_{10}^{2,+}(\Gamma_0(2)) \\
\widetilde{X}_{12,2}(z) &:= X_{12,2}(z) - 2^{11} X_{12,2}(2z) \in \mathcal{QM}_{12}^{2,+}(\Gamma_0(2)) \\
\widetilde{X}_{14,2}(z) &:= X_{14,2}(z) - 2^{13} X_{14,2}(2z) \in \mathcal{QM}_{14}^{2,+}(\Gamma_0(2)).
\end{aligned}$$

One may take a smaller exponent m so that $F(z) - N^m F(Nz)$ becomes completely positive. Such examples can be found when the Fourier expansions of F can be easily expressed in terms of divisor sums or the Ramanujan tau function; see Appendix D for some examples of this type.

6.5 Application: Another proof of the modular form inequality for the Leech lattice packing

By using the positive quasimodular forms constructed in the previous section, we can give an alternative proof of one of the modular form inequalities [19, Lemma A.2] used in the proof of the optimality of the Leech lattice packing.

Recall (see (5.30) and (5.31)) the forms

$$\begin{aligned} F &= 49E_2^2E_4^3 - 25E_2^2E_6^2 - 48E_2E_4^2E_6 - 25E_4^4 + 49E_4E_6^2, \\ G &= H_2^5(2H_2^2 + 7H_2H_4 + 7H_4^2). \end{aligned}$$

Then the inequality [19, Lemma A.2] is equivalent to

$$F(it) < \frac{432}{\pi^2}G(it)$$

for $t > 0$ (5.33). The author gave an algebraic proof of this inequality by proving monotonicity of the quotient $F(it)/G(it)$ (Proposition 5.4.4) and showing that the limit $\lim_{t\to 0^+} F(it)/G(it) = 432/\pi^2$ (Proposition 5.4.6). The monotonicity part was proved by using the differential equations satisfied by F and G, i.e. (5.36) and (5.37). In this section, we give another proof of the monotonicity by using the positive quasimodular forms constructed in the previous section.

It is enough to show that

$$\mathcal{L}_{1,0} := F'G - FG' = (\partial_{14}F)G - F(\partial_{14}G)$$

is positive. Using Sage, one can check that $\mathcal{L}_{1,0}$ can be factored as

$$\mathcal{L}_{1,0} = \frac{105}{8} \cdot H_2^5H_4^2(H_2 + H_4)^2 \cdot L, \tag{6.10}$$

where $L := K_{10}E_2^2 + K_{12}E_2 + K_{14}$ is a quasimodular form of weight 14, level $\Gamma(2)$, and depth 2 with

$$\begin{aligned} K_{10} &= -2(23H_2^4 + 46H_2^3H_4 + 54H_2^2H_4^2 + 16H_2H_4^3 + 8H_4^4)(H_2 + 2H_4), \\ K_{12} &= -2(10H_2^4 + 35H_2^3H_4 + 3H_2^2H_4^2 - 64H_2H_4^3 - 32H_4^4)(H_2^2 + H_2H_4 + H_4^2), \\ K_{14} &= (26H_2^6 + 78H_2^5H_4 + 177H_2^4H_4^2 + 182H_2^3H_4^3 + 51H_2^2H_4^4 - 48H_2H_4^5 - 16H_4^6)(H_2 + 2H_4). \end{aligned}$$

Here K_w (for $w \in \{10, 12, 14\}$) are modular forms of weight w and level $\Gamma(2)$. We observe that each K_w is invariant under $z \mapsto z+1$, so they are in fact modular forms of level $\Gamma_0(2) \subset \Gamma(2)$. The ring of modular forms of level $\Gamma_0(2)$ is generated by two elements $A = H_2^2$ and $B = H_2 + 2H_4$, where both have positive Fourier coefficients:

$$A = 4\left(\sum_{n\geq 0} r_4(2n+1)q^{n+\frac{1}{2}}\right)^2, \quad B = 2\left(1 + \sum_{n\geq 1} r_4(2n)q^n\right).$$

(Here $r_4(n)$ is the number of ways to write n as a sum of four squares.) Let $X^{\langle 2\rangle}_{w,s}(z) := X_{w,s}(2z)$. One can write L as a combination of $X^{\langle 2\rangle}_{8,2}, \widetilde{X}_{8,2}, X^{\langle 2\rangle}_{10,2}, \widetilde{X}_{10,2}, X^{\langle 2\rangle}_{12,2}, \widetilde{X}_{12,2}$ and A, B as follows:

$$L = a_1 X^{\langle 2\rangle}_{8,2}AB + a_2\widetilde{X}_{8,2}AB + a_3 X^{\langle 2\rangle}_{10,2}A + a_4\widetilde{X}_{10,2}A + a_5 X^{\langle 2\rangle}_{12,2}B + a_6\widetilde{X}_{12,2}B \tag{6.11}$$

where all the coefficients $a_1, \dots, a_6$ are positive:

$$\begin{aligned} a_1 &= 78278400, \\ a_2 &= 550800, \\ a_3 &= 90823680, \\ a_4 &= 116640, \\ a_5 &= 678813696000, \\ a_6 &= 331776000, \end{aligned}$$

and the positivity of L follows from the positivity of $X_{8,2}, X_{10,2}, X_{12,2}$ (Proposition 4.4.2) and Example 6.4.4.

6.6 Application: Modular form inequality for universal optimality of the Leech lattice

The last application is a new proof of the inequality (3b) of [20, p. 1067]. In loc. cit., the authors proved the universal optimality of E_8 and the Leech lattice by using Cohn–Kumar's linear programming bound for potential energy [16]. Following the idea of optimal sphere packings [80, 19], they constructed two-variable generating functions of Fourier interpolation bases for dimensions 8 and 24 [20, Theorem 1.7], as Laplace transforms of certain kernel functions. In particular, they proved positivity of the kernels by means of numerical analysis, which is considerably more involved than the methods used in [80, 19].

One of the inequalities is the following. Let

$$\widehat{\mathcal{E}}_{1,0} = 12\pi i(-E_2E_4E_6 + E_6^2 + 720\Delta) \tag{6.12}$$

$$\widehat{\mathcal{E}}_{1,1} = 2\pi^2(E_2^2E_4E_6 - 2E_2E_6^2 - 1728E_2\Delta + E_4^2E_6) \tag{6.13}$$

$$\widehat{\mathcal{E}}_1(\tau) = \tau\widehat{\mathcal{E}}_{1,1}(\tau) + \widehat{\mathcal{E}}_{1,0}(\tau). \tag{6.14}$$

Inequality (3b) of [20, p. 1067] is

$$i\widehat{\mathcal{E}_1}(it) > 0 \tag{6.15}$$

for all $t > 0$. Combined with the other inequalities (1), (2), and (3a) of [20, p. 1067], they proved positivity of the generating function for $d = 24$ and showed the universal optimality of the Leech lattice.

Here, we show that the inequality (6.15) follows from Corollary 6.2.7.

Lemma 6.6.1. The inequality (6.15) is equivalent to the monotonicity of the function $t \mapsto t^{11}X_{12,1}(it)$ on $(0, \infty)$. In particular, it follows from Corollary 6.2.7.

Proof. Recall that (see Table C.1)

$$X_{12,1} = \frac{-12E_2E_4E_6 + 5E_4^3 + 7E_6^2}{3991680} \Leftrightarrow -12E_2E_4E_6 + 5E_4^3 + 7E_6^2 = 3991680X_{12,1}.$$

By Theorem 4.2.1, we have

$$X_{12,1}' = 2X_{6,1}X_{8,1} = 2\cdot\frac{E_2E_4 - E_6}{720}\cdot\frac{-E_2E_6 + E_4^2}{1008} = \frac{11}{3991680}(-E_2^2E_4E_6 + E_2E_4^3 + E_2E_6^2 - E_4^2E_6).$$

Using this, one can express $\widehat{\mathcal{E}}_{1,0}$ and $\widehat{\mathcal{E}}_{1,1}$ in terms of $X_{12,1}$ and $X_{12,1}'$:

$$\widehat{\mathcal{E}}_{1,0} = 12\pi i\left(-E_2E_4E_6 + E_6^2 + \frac{5}{12}(E_4^3 - E_6^2)\right) = \pi i\cdot(-12E_2E_4E_6 + 5E_4^3 + 7E_6^2) = \pi i\cdot 3991680X_{12,1}$$

$$\widehat{\mathcal{E}}_{1,1} = 2\pi^2(E_2^2E_4E_6 - 2E_2E_6^2 - E_2(E_4^3 - E_6^2) + E_4^2E_6) = -\frac{2\pi^2}{11}\cdot 3991680X_{12,1}'$$

and

$$i\widehat{\mathcal{E}}_1(it) = -t\widehat{\mathcal{E}}_{1,1}(it) + i\widehat{\mathcal{E}}_{1,0}(it) = 3991680\pi\left(\frac{2\pi t}{11}X_{12,1}'(it) - X_{12,1}(it)\right)$$

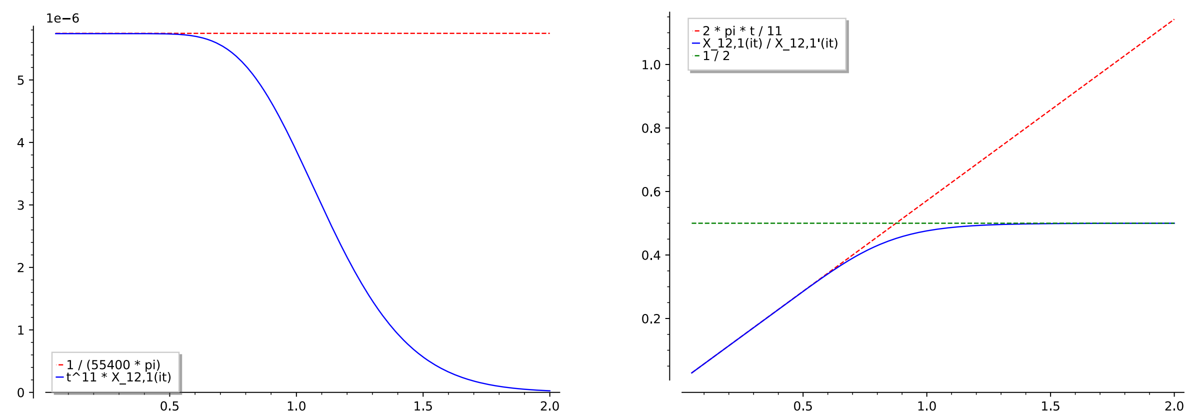
$$= -362880\pi t^{-10} \cdot \frac{\mathrm{d}}{\mathrm{d}t}(t^{11}X_{12,1}(it))$$

which proves the claim. □

Figure 6.4 shows the graphs of $t \mapsto t^{11}X_{12,1}(it)$ and $t \mapsto X_{12,1}(it)/X'_{12,1}(it)$, where the former is monotone decreasing and the latter is monotone increasing. By (6.4), we have

$$\lim_{t\to 0^+} t^{11}X_{12,1}(it) = \frac{6(-1)^{\frac{12}{2}}\beta_{10}}{\pi} = \frac{1}{55400\pi}$$

and the tangent line of the latter at $t = 0$ is $l(t) := \frac{2\pi t}{11}$. We leave it as future work to give simpler proofs of the other inequalities (1), (2), and (3a) of [20, p. 1067].



(a) $t^{11}X_{12,1}(it)$ (b) $X_{12,1}(it)/X'_{12,1}(it)$

Figure 6.4: Graphs of $t^{11}X_{12,1}(it)$ and $X_{12,1}(it)/X'_{12,1}(it)$ with asymptotes as $t \to 0^+$ and $t \to \infty$.

Chapter 7

Extremal quasimodular forms of higher levels

In Chapter 4, we studied the positivity of level 1 extremal quasimodular forms of Kaneko–Koike [40]. In this chapter, we study analogues of their results for extremal quasimodular forms of level $\Gamma_0(N)$ for $N = 2, 3, 4$, introduced by Sakai and Tsutsumi [67]. We first prove uniqueness of extremal quasimodular forms of level $\Gamma_0(N)$ and depth s for $(N, s) \in \{(2,1), (2,2), (3,1), (4,1)\}$ by adapting Pellarin's argument [61] (Corollary 7.2.4).

For each pair (N, s) listed above, we derive new recurrence relations (Theorems 7.3.7 and 7.4.5 for depth 1, and Proposition 7.3.16 for depth 2), and use them to prove positivity of the depth 1 forms $\mathcal{D}_w$ and $\mathcal{F}_w$ (Propositions 7.3.8 and 7.5.3). In all cases, we also classify those with integral Fourier coefficients (Theorems 7.3.14, 7.4.7, and 7.3.19), extending the level 1 results of Kaminaka–Kato [39] and Nakaya [57]. For $(N, s) = (2, 2)$, we further classify the completely positive forms (Corollary 7.3.17), showing that only finitely many weights give completely positive forms, in contrast to the depth 1 case.

7.1 Extremal quasimodular forms à la Sakai–Tsutsumi

It is natural to consider extremal quasimodular forms of higher levels. In [67], Sakai and Tsutsumi defined them for level $\Gamma_0(N)$ with $N \geq 1$ (possibly with nontrivial characters) in the same way.

Definition 7.1.1 (Sakai–Tsutsumi [67]). Let $f = \sum_{n\geq 0} a_n q^n$ be a quasimodular form of weight w and depth r with a character χ of $\Gamma_0(N)$. Let

$$m = \dim \mathcal{QM}_w^r(\Gamma_0(N), \chi) = \sum_{i=0}^{r} \dim \mathcal{M}_{w-2i}(\Gamma_0(N), \chi).$$

Then f is an extremal quasimodular form if its first m Fourier coefficients satisfy

$$a_0 = a_1 = \cdots = a_{m-2} = 0, \quad a_{m-1} \neq 0.$$

We say that f is normalized if $a_{m-1} = 1$.

They studied extremal quasimodular forms of depth 1 and level $N = 2, 3, 4$, constructing explicit families via recurrence relations that also satisfy certain modular linear differential equations.

7.2 Uniqueness of extremal quasimodular forms

We prove uniqueness (up to normalization) of extremal quasimodular forms of even weight, depth 1, trivial character, and level $\Gamma_0(N)$ for $N = 2, 3, 4$, by adapting Pellarin's argument [61]. Note that Lin and Yang used the same argument to prove uniqueness of extremal forms in Atkin–Lehner subspaces of level $\Gamma_0^+(2)$ and $\Gamma_0^+(3)$, where $\Gamma_0^+(N)$ is a subgroup of $\mathrm{SL}_2(\mathbb{R})$ generated by $\Gamma_0(N)$ and Atkin–Lehner involutions [52, Corollary 2.16].

The main idea of [61] is to consider the following quantity

$$\kappa_s(w) := \dim \mathcal{M}_{(s+1)w}(\mathrm{SL}_2(\mathbb{Z})) - \dim \mathcal{QM}_w^{\leq s}(\mathrm{SL}_2(\mathbb{Z})) \in \mathbb{Z} \tag{7.1}$$

and prove the following estimate:

Theorem 7.2.1 (Theorem 2.3 of [61]). Let $\delta_s(w) := \dim \mathcal{QM}_w^{\leq s}(\mathrm{SL}_2(\mathbb{Z}))$. We have

$$\delta_s(w) - 1 \leq \nu_{\max}(s, w) \leq \delta_s(w) - 1 + \kappa_s(w), \tag{7.2}$$

where $\nu_{\max}(s, w)$ is the largest possible vanishing order at infinity,

$$\nu_{\max}(s, w) := \max\{n : \exists f \in \mathcal{QM}_w^{\leq s}(\mathrm{SL}_2(\mathbb{Z}))\backslash\{0\}, f = q^n + O(q^{n+1})\}. \tag{7.3}$$

In particular, $\kappa_s(w) = 0$ for $s = 1, 2, 3, 4$ [61, Lemma 2.1], and combined with uniqueness of *extremal modular forms* (i.e. depth 0) of level 1, this proves the uniqueness of level 1 extremal forms of depths 1 through 4. Note that $\mathcal{M}_w(\mathrm{SL}_2(\mathbb{Z}))$ has a basis $g_0, g_1, \ldots, g_{d-1}$ of the form $g_j(z) = \sum_{n \geq j} a_n(g_j) q^n$ [74, p. 20], which proves uniqueness of extremal modular forms of level 1.

For higher levels, it is natural to consider the following generalization of (7.1):

$$\delta_s(w, N) := \dim \mathcal{QM}_w^{\leq s}(\Gamma_0(N)) \tag{7.4}$$

$$\begin{aligned}\kappa_s(w, N) &:= \dim \mathcal{M}_{(s+1)w}(\Gamma_0(N)) - \dim \mathcal{QM}_w^{\leq s}(\Gamma_0(N)) \\ &= \delta_0((s+1)w, N) - \delta_s(w, N).\end{aligned} \tag{7.5}$$

Following the arguments in [61, 52], it suffices to establish the following.

Proposition 7.2.2. For each s and N, $\kappa_s(w, N)$ is increasing in w and independent of w for sufficiently large w, i.e. $\kappa_s(N) := \lim_{w \to \infty} \kappa_s(w, N)$ exists. Moreover, $\kappa_1(2) = \kappa_2(2) = \kappa_1(3) = \kappa_1(4) = 0$.

Proof. For $N = 2$, one has

$$\delta_0(w, 2) = \dim \mathcal{M}_w(\Gamma_0(2)) = \left\lfloor \frac{w}{4} \right\rfloor + 1. \tag{7.6}$$

Hence $\delta_0(w+4, 2) = \delta_0(w, 2) + 1$ holds for all w, and if $w \geq 2s + 4$, we can write $w = w' + 4$ with $w' - 2i \geq 0$ for all $i \leq s$, so

$$\begin{aligned}\kappa_s(w, 2) &= \delta_0((s+1)w, 2) - \delta_s(w, 2) \\ &= (s+1) + \delta_0((s+1)w', 2) - (s+1) - \sum_{i=0}^{s} \delta_0(w' - 2i, 2) \\ &= \delta_0((s+1)w', 2) - \sum_{i=0}^{s} \delta_0(w' - 2i, 2) \\ &= \kappa_s(w', 2)\end{aligned}$$

and this also proves monotonicity. $\kappa_1(2) = \kappa_2(2) = 0$ are equivalent to

$$\begin{aligned}\delta_0(2w, 2) &= \delta_0(w, 2) + \delta_0(w-2, 2), \\ \delta_0(3w, 2) &= \delta_0(w, 2) + \delta_0(w-2, 2) + \delta_0(w-4, 2),\end{aligned}$$

which can be proved using (7.6). Similarly, the dimensions of even-weight modular forms of level 3 and 4 are

$$\delta_0(w,3) = \dim \mathcal{M}_w(\Gamma_0(3)) = \left\lfloor \frac{w}{3} \right\rfloor + 1, \tag{7.7}$$

$$\delta_0(w,4) = \dim \mathcal{M}_w(\Gamma_0(4)) = \frac{w}{2} + 1, \tag{7.8}$$

A similar argument with $\delta_0(w,3) = \delta_0(w-6,3) + 2$ and $\delta_0(w,4) = \delta_0(w-2,4) + 1$ proves $\kappa_s(w,3) = \kappa_s(w-6,3)$ and $\kappa_s(w,4) = \kappa_s(w-2,4)$. One can get monotonicity of $\kappa_s(w,3)$ and $\kappa_s(w,4)$ from these, and also

$$\kappa_1(3) = 0 \Leftrightarrow \delta_0(2w,3) = \delta_0(w,3) + \delta_0(w-2,3),$$
$$\kappa_1(4) = 0 \Leftrightarrow \delta_0(2w,4) = \delta_0(w,4) + \delta_0(w-2,4),$$

can be shown using (7.7) and (7.8). □

Proposition 7.2.3. For each $w \in 2\mathbb{Z}$ and $N = 2, 3, 4$, there exists a unique extremal modular form of weight w and level $\Gamma_0(N)$ up to scalar multiples.

See Theorems 7.3.3 and 7.4.2 for the level 2 and 3 cases; the level 4 case is discussed in Section 7.5.

Corollary 7.2.4. For

$$(N,s) \in \{(2,1),(2,2),(3,1),(4,1)\} \tag{7.9}$$

and even w, extremal quasimodular forms of weight w, depth s, and level $\Gamma_0(N)$ are unique up to scalar multiples.

Proof. One can show that an inequality similar to (7.2) holds for general levels:

$$\delta_s(w,N) - 1 \le \nu_{\max}(s,w,N) \le \delta_s(w,N) - 1 + \kappa_s(w,N) \tag{7.10}$$

where

$$\nu_{\max}(s,w,N) := \max\{n : \exists f \in Q\mathcal{M}_w^{\le s}(\Gamma_0(N))\backslash\{0\}, f = q^n + O(q^{n+1})\}. \tag{7.11}$$

Now, Proposition 7.2.2 shows $\delta_s(w,N) - 1 = \nu_{\max}(s,w,N)$ and completes the proof. □

Note that $\kappa_s(N) > 0$ for other (N,s) with $N = 2,3,4$, so the argument does not apply in general. However, we can still check the uniqueness of extremal forms of small depths and weights by considering the first few Fourier coefficients of a basis of $\mathcal{QM}_w^{\leq s}(\Gamma_0(N))$. Let $f_1, \cdots, f_d$ be a basis of $\mathcal{QM}_w^{\leq s}(\Gamma_0(N))$ with Fourier expansions

$$f_i(z) = \sum_{n\geq 0} a_{i,n} q^n,$$

which can be computed using a basis of $\mathcal{M}_{w'}(\Gamma_0(N))$ for $w' \leq w$ and E_2. Then the uniqueness of an extremal quasimodular form of weight w, depth s, and level $\Gamma_0(N)$ is equivalent to the invertibility of the $d \times d$ matrix

$$\begin{pmatrix} a_{1,0} & a_{1,1} & \cdots & a_{1,d-1} \\ a_{2,0} & a_{2,1} & \cdots & a_{2,d-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d,0} & a_{d,1} & \cdots & a_{d,d-1} \end{pmatrix}.$$

Using Sage, we checked this for $(N, s) = (2,3), (2,4), (3,2), (3,3), (3,4)$ and $w \leq 50$. Based on our calculation, we conjecture that uniqueness holds for all even weights with these levels and depths. Details can be found in Appendix A.2.

Conjecture 7.2.5. For each $N \geq 1$, even $w \geq 2$, and $s \geq 0$, there exists a unique normalized extremal quasimodular form of weight w, depth s, and level $\Gamma_0(N)$.

7.3 Level $\Gamma_0(2)$

Define the following modular forms of level $\Gamma_0(2)$:[1]

$$A_2(z) = 2E_2(2z) - E_2(z) = 1 + 24q + 24q^2 + 96q^3 + 24q^4 + 144q^5 + \cdots, \tag{7.12}$$

$$A_{4,0}(z) = E_4(2z) = 1 + 240q^2 + 2160q^4 + 6720q^6 + 17520q^8 + 30240q^{10} + \cdots, \tag{7.13}$$

$$A_{4,1}(z) = \frac{\eta(2z)^{16}}{\eta(z)^8} = \frac{A_2(z)^2 - A_{4,0}(z)}{48} = q + 8q^2 + 28q^3 + 64q^4 + 126q^5 + \cdots. \tag{7.14}$$

These are modular forms of weights $2, 4, 4$, and the ring of quasimodular forms of level $\Gamma_0(2)$ is generated by these together with E_2:

$$\mathcal{QM}(\Gamma_0(2)) = \mathbb{C}[E_2, A_2, A_{4,0}] = \mathbb{C}[E_2, A_2, A_{4,1}].$$

Note that $A_{4,1}(z)$ is not a cusp form in the usual sense, since it only vanishes at one cusp (∞).

[1] A_2 and $A_{4,1}$ are denoted by H_2 and Δ_2 in [67], respectively.

Lemma 7.3.1. A_2, $A_{4,0}$, and $A_{4,1}$ have positive and integral Fourier coefficients.

Proof. The Fourier expansion of A_2 is

$$\begin{aligned} A_2(z) &= 2\left(1 - 24\sum_{n\geq 1}\sigma_1(n)q^{2n}\right) - \left(1 - 24\sum_{n\geq 1}\sigma_1(n)q^n\right) \\ &= 1 + 24\sum_{n\geq 1}\left(\sigma_1(n) - 2\sigma_1\left(\frac{n}{2}\right)\right)q^n \end{aligned}$$

where we define σ_1 to be zero for non-integer inputs. For each n, write $n = 2^k m$ with $k \geq 0$ and odd m; then one can show that

$$\sigma_1(n) - 2\sigma_1\left(\frac{n}{2}\right) = \sigma_1(m) > 0.$$

Positivity of Fourier coefficients of $A_{4,0}(z) = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^{2n}$ is clear. For $A_{4,1}$, we can write it as

$$\begin{aligned} A_{4,1} &= \frac{A_2'}{24} - \frac{E_2'(2z)}{12} + \left(\frac{E_2(2z) - E_2(z)}{12}\right)^2 \qquad (7.15) \\ &= \sum_{n\geq 1} n\left(\sigma_1(n) - \sigma_1\left(\frac{n}{2}\right)\right)q^n + 4\left[\sum_{n\geq 1}\left(\sigma_1(n) - \sigma_1\left(\frac{n}{2}\right)\right)q^n\right]^2 \end{aligned}$$

and the claim follows. □

Remark 7.3.2. In [44, p. 153], it is shown that the square root of $A_{4,1}$ has positive integral Fourier coefficients:

$$\frac{\eta(2z)^8}{\eta(z)^4} = \sum_{n\geq 1, 2\nmid n}\sigma_1(n)q^{\frac{n}{2}} = q^{\frac{1}{2}} + 4q^{\frac{3}{2}} + 6q^{\frac{5}{2}} + 8q^{\frac{7}{2}} + 13q^{\frac{9}{2}} + \cdots . \qquad (7.16)$$

Also, the eighth root of $A_{4,1}$ is

$$\frac{\eta(2z)^2}{\eta(z)} = \frac{q^{\frac{1}{6}}\prod_{n\geq 1}(1-q^{2n})^2}{q^{\frac{1}{24}}\prod_{n\geq 1}(1-q^n)} = q^{\frac{1}{8}}\sum_{n\geq 1}c_2(n)q^n = q^{\frac{1}{8}}\sum_{n\geq 0}q^{\binom{n+1}{2}}$$

where $c_2(n)$ is the number of 2-core partitions of n [29, 33, 4].

From Lemma 7.3.1, we can prove the following analogue of the Kaneko–Koike conjecture for "depth 0" extremal forms of level $\Gamma_0(2)$.

Theorem 7.3.3. Normalized extremal modular forms of weight w and level $\Gamma_0(2)$ are unique and have positive and integral Fourier coefficients.

Proof. From $\dim_{\mathbb{C}} \mathcal{M}_w(\Gamma_0(2)) = \lfloor \frac{w}{4} \rfloor + 1$, the extremal modular form of level $\Gamma_0(2)$ has vanishing order $\lfloor \frac{w}{4} \rfloor$ at the cusp. Hence, it should be of the form $A_2^a A_{4,1}^b$ where

$$(a,b) = \begin{cases} (0, \frac{w}{4}) & w \equiv 0 \pmod 4 \\ (1, \frac{w-2}{4}) & w \equiv 2 \pmod 4 \end{cases} \tag{7.17}$$

so that $2a + 4b = w$ and $b = \lfloor \frac{w}{4} \rfloor$. Then the claim follows from Lemma 7.3.1. □

Remark 7.3.4. The analogue of Theorem 7.3.3 fails at level 1: extremal modular forms of level 1 are of the form $E_4^a E_6^b \Delta^c$, and their Fourier coefficients are not positive in general.

Remark 7.3.5. One can construct a Miller basis of $\mathcal{M}_w(\Gamma_0(2))$ using A_2 and $A_{4,1}$, i.e. a basis $g_0, g_1, \dots, g_{\lfloor \frac{w}{4} \rfloor}$ where the i-th coefficient of g_j is $a_i(g_j) = \delta_{ij}$ when $0 \le i, j \le \lfloor \frac{w}{4} \rfloor$ (see [74, p. 20] for the case of level 1), by taking linear combinations of $A_2^a A_{4,1}^b$ above. Integrality of the coefficients implies that integral linear combinations of the elements of the basis are precisely the modular forms of level $\Gamma_0(2)$ with integral Fourier coefficients. See Appendix A.2 for more details.

Depth 1

Now we consider the depth 1 extremal forms. Let $\mathcal{D}_w = \mathcal{D}_{w,1}$ denote the (unique) normalized extremal quasimodular form of weight w, depth 1, and level $\Gamma_0(2)$.[2] Define

$$E_2^{[2]}(z) := \frac{E_2(z) + 2A_2(z)}{3} = (\log A_{4,1}(z))' = 1 + 8 \sum_{n \ge 1} \left(\sigma_1(n) - 4\sigma_1\left(\frac{n}{2}\right) \right) q^n \tag{7.18}$$

which will play the role of E_2 in the case of level $\Gamma_0(2)$.[3] We also denote the *level* $\Gamma_0(2)$ *Serre derivative* as

$$\partial_k^{[2]} F := F' - \frac{k}{4} E_2^{[2]} F = \partial_k F - \frac{k}{6} A_2 F. \tag{7.19}$$

It is known that $\mathcal{D}_w$ satisfies the following modular differential equation [67, Theorem A]:

$$\mathcal{D}_w'' - \frac{w}{2} E_2^{[2]} \mathcal{D}_w' + \frac{w(w-1)}{4} E_2^{[2]\prime} \mathcal{D}_w = 0. \tag{7.20}$$

[2] In [67], the authors used $\mathcal{D}_n$ for weight $2n$ extremal forms, but we use this notation so that all the subscripts stand for weights of quasimodular forms, like (7.12) or (7.13), (7.14).

[3] It is denoted by $E_2^{(2)}$ in [67], while we use [2] as a superscript to avoid confusion with second derivatives.

Using this, we can obtain a recurrence relation for the Fourier coefficients of $\mathcal{D}_w$.

Proposition 7.3.6. Write the Fourier expansion of $\mathcal{D}_w$ as

$$\mathcal{D}_w = \sum_{n \geq \frac{w}{2}} d_n^{(w)} q^n = \sum_{k \geq 0} f_k(w) q^{\frac{w}{2}+k} \tag{7.21}$$

with $f_k(w) = d^{(w)}_{\frac{w}{2}+k}$. Then $f_k(w)$ is a rational function in w, and it satisfies the following identity:

$$f_k(w) = \frac{1}{\left(\frac{w}{2}+k\right)k} \left(\sum_{i=1}^{k} \left(\frac{w}{2}\left(\frac{w}{2}+(k-i)\right) - \frac{w(w-1)i}{4} \right) b_i f_{k-i}(w) \right) \tag{7.22}$$

where $f_0(w) = 1$ and b_i is the i-th Fourier coefficient of $E_2^{[2]}$.

Proof. Compare the coefficients of $q^{\frac{w}{2}+k}$ in (7.20). □

Here is a list of $f_k(w)$ for $1 \leq k \leq 4$:

$$\begin{aligned} f_1(w) &= \frac{4w}{w+2}, \\ f_2(w) &= \frac{2w(w^2+12w-4)}{(w+2)(w+4)}, \\ f_3(w) &= \frac{8w(w^3+14w^2-16w+16)}{(w+2)(w+4)(w+6)}, \\ f_4(w) &= \frac{2w(w^5+\frac{63}{2}w^4+312w^3-654w^2+1472w-480)}{(w+2)(w+4)(w+6)(w+8)}. \end{aligned} \tag{7.23}$$

Theorem 7.3.7. For $w \geq 2$, let $\mathcal{D}_w$ be the normalized extremal quasimodular form of weight w and depth 1 defined in [67]. Then these forms satisfy the following recurrence relations:

$$\mathcal{D}_{w+4} = \frac{(w+2)(w+4)}{16(w+3)^2}(A_2\mathcal{D}_{w+2} - A_{4,1}\mathcal{D}_w). \tag{7.24}$$

$$\mathcal{D}_{w+2} = \frac{w+2}{8(w+1)^2}\left(\frac{5w+1}{12}A_2\mathcal{D}_w - \partial_{w-1}\mathcal{D}_w\right) \tag{7.25}$$

$$\mathcal{D}'_{w+2} = -4(w+1)\mathcal{D}_2\mathcal{D}_{w+2} + \left(\frac{w+2}{2}\right)A_{4,1}\mathcal{D}_w \tag{7.26}$$

Proof. (7.24) corrects a minor error in [67, Lemma 1]. For (7.25), one can check that the right-hand side is a normalized quasimodular form of weight $w+2$ and depth 1 with vanishing order $\frac{w}{2}+1$ at the cusp, and Corollary 7.2.4 implies that it must be $\mathcal{D}_{w+2}$. For the last identity, differentiating $(7.25)_w$ gives

$$\begin{aligned}\mathcal{D}'_{w+2} &= \frac{w+2}{8(w+1)^2}\left(\frac{5w+1}{12}A_2\mathcal{D}'_w + \frac{5w+1}{12}A'_2\mathcal{D}_w - \mathcal{D}''_w + \frac{w-1}{12}E'_2\mathcal{D}_w + \frac{w-1}{12}E_2\mathcal{D}'_w\right)\\ &= \frac{w+2}{8(w+1)}\left(4\mathcal{D}_2\mathcal{D}'_w + \frac{1}{12}((2w+1)A'_2 + (w-1)E'_2)\mathcal{D}_w\right)\end{aligned}$$

where we used (7.20) to rewrite $\mathcal{D}''_w$. Now, add this to $4(w+1)\mathcal{D}_2\mathcal{D}_{w+2}$ with $(7.25)_w$; then the $\mathcal{D}_2\mathcal{D}'_w$ term vanishes and we get $(7.26)_w$. □

Proposition 7.3.8. For $w \geq 2$, $\mathcal{D}_w$ is positive, i.e. $\mathcal{D}_w(it) > 0$ for all $t > 0$.

Proof. Define $\widetilde{\mathcal{D}}_w$ as

$$\widetilde{\mathcal{D}}_w(z) := \left(\frac{\eta(2z)}{\eta(z)^2}\right)^{2(w-1)}\mathcal{D}_w(z). \tag{7.27}$$

We will show that $\widetilde{\mathcal{D}}_w(z)$ is completely positive, and the positivity of $\mathcal{D}_w(z)$ will follow directly from the product formula of η. For $w=2$,

$$\mathcal{D}_2 = \frac{A_2(z)-E_2(z)}{48} = \frac{E_2(2z)-E_2(z)}{24} = \sum_{n\geq 1}\left(\sigma_1(n) - \sigma_1\left(\frac{n}{2}\right)\right)q^n \tag{7.28}$$

is completely positive, and also

$$\frac{\eta(2z)}{\eta(z)^2} = \prod_{n\geq 1}\frac{(1-q^{2n})}{(1-q^n)^2} = \prod_{n\geq 1}\frac{1+q^n}{1-q^n} = \prod_{n\geq 1}(1+q^n)(1+q^n+q^{2n}+\cdots) \tag{7.29}$$

is completely positive as well, hence $\widetilde{\mathcal{D}}_2(z)$ is completely positive. By $\eta' = \frac{1}{24}E_2\eta$ and $(7.26)_{w-2}$, the derivative of $\widetilde{\mathcal{D}}_w$ is

$$\begin{aligned}\widetilde{\mathcal{D}}'_w(z) &= 2(w-1)\left(\frac{\eta(2z)}{\eta(z)^2}\right)^{2w-3}\frac{\frac{1}{12}E_2(2z)\eta(2z)\eta(z)^2 - \frac{1}{12}E_2(z)\eta(2z)\eta(z)^2}{\eta(z)^4}\mathcal{D}_w(z)\\ &\quad + \left(\frac{\eta(2z)}{\eta(z)^2}\right)^{2(w-1)}\mathcal{D}'_w(z)\\ &= \left(\frac{\eta(2z)}{\eta(z)^2}\right)^{2(w-1)}[4(w-1)\mathcal{D}_2(z)\mathcal{D}_w(z) + \mathcal{D}'_w(z)]\end{aligned}$$

$$= \frac{w}{2} A_{4,1}(z) \left(\frac{\eta(2z)}{\eta(z)^2}\right)^{2(w-1)} \mathcal{D}_{w-2}(z)$$
$$= \frac{w}{2} A_{4,1}(z) \left(\frac{\eta(2z)^2}{\eta(z)^4}\right) \widetilde{\mathcal{D}}_{w-2}(z).$$

By induction, $\widetilde{\mathcal{D}}_w(z)$ is completely positive for all w (recall Lemma 7.3.1 and (7.29)). □

Lemma 7.3.9. For $N \in \mathbb{Z}_{\geq 2}$, $N^3 E_2'(Nz) - E_2'(z) \in \mathcal{QM}_4^{\leq 1}(\Gamma_0(N))$ is positive.

Proof. This follows from Lemma 6.1.2 and Corollary 6.4.1. □

Corollary 7.3.10. For any $N \in \mathbb{Z}_{\geq 2}$ and $t > 0$, $N^2 E_2(iNt) - E_2(it) > N^2 - 1$.

Proof. By Lemma 7.3.9, $t \mapsto N^2 E_2(iNt) - E_2(it)$ is decreasing, hence $N^2 E_2(iNt) - E_2(it) > \lim_{t\to\infty} N^2 E_2(iNt) - E_2(it) = N^2 - 1$. □

Proposition 7.3.11. For $w \geq 2$, $\mathcal{D}_w'$ is positive.

Proof. From (7.18), one can check that the differential equation (7.20) is equivalent to

$$\left(\frac{\mathcal{D}_w'}{A_{4,1}^{\frac{w}{2}}}\right)' = -\frac{w(w-1)}{4} E_2^{[2]\prime} \frac{\mathcal{D}_w}{A_{4,1}^{\frac{w}{4}}}. \tag{7.30}$$

By Lemma 7.3.9 with $N = 2$, $E_2^{[2]\prime} > 0$ and the right-hand side of (7.30) is negative, hence $t \mapsto \mathcal{D}_w'(it)/A_{4,1}(it)^{\frac{w}{2}}$ is a monotone increasing function on $t > 0$. Thus it is enough to show that the limit $\lim_{t\to 0^+} \mathcal{D}_w'(it)/A_{4,1}(it)^{\frac{w}{2}}$ is zero. Using the transformation law of $\eta(z)$, we get

$$A_{4,1}\left(\frac{i}{2t}\right) = \frac{t^4}{16}\frac{\eta(it)^{16}}{\eta(2it)^8} = \frac{t^4}{16}\prod_{n\geq 1}\left(\frac{1-e^{-2\pi nt}}{1+e^{-2\pi nt}}\right)^8 = \frac{t^4}{16}(1 + O(e^{-2\pi t})) \tag{7.31}$$

as $t \to \infty$. For $\mathcal{D}_w'(it)$, write $\mathcal{D}_w' = F_0 + F_1 E_2 + F_2 E_2^2$ with $F_j \in \mathcal{M}_{w-2j}(\Gamma_0(2))$. Then

$$z^{-w}\mathcal{D}_w'\left(-\frac{1}{2z}\right) = z^{-w} F_0\left(-\frac{1}{2z}\right) + z^{-(w-2)} F_1\left(-\frac{1}{2z}\right) \cdot z^{-2} E_2\left(-\frac{1}{2z}\right) + z^{-(w-4)} F_2\left(-\frac{1}{2z}\right) \cdot z^{-4} E_2\left(-\frac{1}{2z}\right)^2$$

$$= \widetilde{F}_1(z) + \widetilde{F}_2(z)\left(4E_2(2z) - \frac{12\pi i}{z}\right) + \widetilde{F}_3(z)\left(4E_2(2z) - \frac{12\pi i}{z}\right)^2$$

for some $\widetilde{F}_j \in \mathcal{M}_{w-2j}(\Gamma_0(2))$ (these are multiples of Atkin–Lehner involutions of F_j). Considering $z = it$ we get $\mathcal{D}'_w\left(\frac{i}{2t}\right) = O(t^w)$ as $t \to \infty$ (note that $\widetilde{F}_j(it) = O(1)$), and combining with (7.31) gives

$$\lim_{t\to 0^+} \frac{\mathcal{D}'_w(it)}{A_{4,1}(it)^{\frac{w}{2}}} = \lim_{t\to\infty} \frac{\mathcal{D}'_w\left(\frac{i}{2t}\right)}{A_{4,1}\left(\frac{i}{2t}\right)^{\frac{w}{2}}} = \lim_{t\to\infty} \frac{O(t^w)}{\frac{t^{2w}}{2^{2w}}(1 + O(e^{-2\pi t}))} = 0.$$

□

Propositions 7.3.8 and 7.3.11 show that the function $t \mapsto \mathcal{D}_w(it)$ is positive and monotone decreasing for all $t > 0$. In fact, we conjecture that $\mathcal{D}_w$ satisfies the analogue of the Kaneko–Koike conjecture.

Conjecture 7.3.12. For all even $w \geq 2$, $\mathcal{D}_w$ is completely positive.

Although we cannot prove this, Propositions 7.3.8 and 7.3.11 provide some evidence for this conjecture in view of Proposition 3.2.4, where complete positivity is equivalent to the positivity of all higher derivatives.

Instead, Lemma 7.3.1 and (7.24) immediately imply the following result, which is an analogue of other parts of the Kaneko–Koike conjecture.

Corollary 7.3.13. For even $w \geq 2$, the prime factors of the denominators of all the Fourier coefficients of $\mathcal{D}_w$ are smaller than the weight w.

By observing the second nonzero coefficient of $\mathcal{D}_w$, we can easily determine all extremal forms with integral Fourier coefficients, as was done for the level 1 and depth ≤ 4 cases in [39] and [57].

Theorem 7.3.14. $\mathcal{D}_w$ has an integral Fourier expansion if and only if $w = 2$.

Proof. By (7.23), $\mathcal{D}_w$ has integral Fourier coefficients only if $w + 2$ divides 8, i.e. $w = 2$ or $w = 6$. $\mathcal{D}_2$ has integral Fourier coefficients from (7.28), but the coefficient of q^5 in $\mathcal{D}_6$ is $\frac{78}{5}$ (Table C.2). □

Depth 2

We proved uniqueness of the (normalized) depth 2 extremal quasimodular forms in Corollary 7.2.4, and we will denote the weight w forms by $\mathcal{D}_{w,2}$. It is natural to ask whether they satisfy differential equations and recurrence relations similar to the ones for level 1 [40, 32]. In the case of level 1 and depth $s \le 4$, the extremal quasimodular forms $X_{w,s}$ are solutions of the differential equation

$$X_{w,s}^{(s+1)} - \frac{w}{12}[E_2, X_{w,s}]_s^{(2,w-s)} = 0$$

and one can try to find a similar equation for $\mathcal{D}_{w,2}$. In fact, direct computation shows that the differential equation (7.20) for $\mathcal{D}_w = \mathcal{D}_{w,1}$ is equivalent to

$$\mathcal{D}_{w,1}'' - \frac{w}{4}[E_2^{[2]}, \mathcal{D}_{w,1}]_1^{(2,w-1)} = 0.$$

An analogous equation holds for depth 2 as well.

Theorem 7.3.15. For $w \equiv 0 \pmod 4$, the extremal quasimodular form of level $\Gamma_0(2)$, weight w, and depth 2 satisfies the following differential equation:

$$\mathcal{D}_{w,2}''' - \frac{w}{4}[E_2^{[2]}, \mathcal{D}_{w,2}]_2^{(2,w-2)} \tag{7.32}$$

$$= \mathcal{D}_{w,2}''' - \frac{3w}{4}E_2^{[2]}\mathcal{D}_{w,2}'' + \frac{3(w-1)w}{4}E_2^{[2]\prime}\mathcal{D}_{w,2}' - \frac{(w-2)(w-1)w}{8}E_2^{[2]\prime\prime}\mathcal{D}_{w,2} = 0, \tag{7.33}$$

or equivalently,

$$\partial_{w+2}^{[2]}\partial_w^{[2]}\partial_{w-2}^{[2]}\mathcal{D}_{w,2} - \left(\frac{-A_2^2 + 4A_{4,0}}{3}\right)\left[\frac{3w^2-4}{16}\partial_{w-2}^{[2]}\mathcal{D}_{w,2} + \frac{(w+1)(w-2)^2}{32}A_2\mathcal{D}_{w,2}\right] = 0 \tag{7.34}$$

Proof. We follow the argument in [32]. One can directly check that $\mathcal{D}_{4,2}$ satisfies the differential equation. Define the operator K_w^{up} as

$$K_w^{\mathrm{up}} f = \partial_w^{[2]}\partial_{w-2}^{[2]} f + \alpha_w^{(1)} A_2 \partial_{w-2}^{[2]} f + \alpha_w^{(2)} A_2^2 f + \alpha_w^{(3)} A_{4,1} f \tag{7.35}$$

where

$$\alpha_w^{(1)} = -\frac{(w+2)(3w^2+12w+10)}{3(7w^2+28w+24)}$$

$$\alpha_w^{(2)} = -\frac{(w+1)(15w^3+48w^2+4w-40)}{12(7w^2+28w+24)}$$
$$\alpha_w^{(3)} = \frac{16(w+1)(16w^3+55w^2+20w-28)}{3(7w^2+28w+24)}$$

By using induction, we can prove that if f satisfies $(7.32)_w$, then $K_w^{\mathrm{up}} f$ satisfies $(7.32)_{w+4}$. More precisely, we can express $\partial_{w+2}^{[2]} K_w^{\mathrm{up}} \mathcal{D}_{w,2}$, $\partial_{w+4}^{[2]}\partial_{w+2}^{[2]} K_w^{\mathrm{up}} \mathcal{D}_{w,2}$ and $\partial_{w+6}^{[2]}\partial_{w+4}^{[2]}\partial_{w+2}^{[2]} K_w^{\mathrm{up}} \mathcal{D}_{w,2}$ as a linear combination of $\partial_w^{[2]}\partial_{w-2}^{[2]}\mathcal{D}_{w,2}$, $\partial_{w-2}^{[2]}\mathcal{D}_{w,2}$, and $\mathcal{D}_{w,2}$ whose coefficients are modular forms of level $\Gamma_0(2)$. Then $(7.34)_{w+4}$ for $f = K_w^{\mathrm{up}}\mathcal{D}_{w,2}$ becomes identically zero. □

A dimension computation shows that the vanishing order of $\mathcal{D}_{w,2}$ is $\lfloor \frac{3w}{4} \rfloor$. By comparing coefficients of $q^{\frac{3w}{4}+1}$ and $q^{\frac{3w}{4}+2}$ of (7.33), for $w \equiv 0 \pmod 4$ the q-expansion of $\mathcal{D}_{w,2}$ starts with

$$\begin{aligned}\mathcal{D}_{w,2} &= q^{\frac{3w}{4}} - \frac{2w(w^2-12w-16)}{(3w+4)^2} q^{\frac{3w}{4}+1} \\ &\quad + \frac{w(2w^5+21w^4+1136w^3+4048w^2+2432w-1024)}{(3w+4)^2(3w+8)^2} q^{\frac{3w}{4}+2} + \cdots\end{aligned} \tag{7.36}$$

We can also use this to compare the second nonzero coefficients of $\mathcal{D}_{w+4,2}$ and $K_w^{\mathrm{up}}\mathcal{D}_{w,2}$ and obtain the constant of proportionality:

$$K_w^{\mathrm{up}}\mathcal{D}_{w,2} = \frac{2^{20}(w+1)^3(w+2)^4(w+3)^3}{3^3(w+4)^2(3w+4)^2(3w+8)^2(7w^2+28w+24)}\mathcal{D}_{w+4,2}.$$

Proposition 7.3.16. For $w \equiv 0 \pmod 4$, we have the following recurrence relation:

$$\mathcal{D}_{w+2,2} = \frac{(3w+4)^2}{256(w+1)^3}\left((w+1)A_2\mathcal{D}_{w,2} - 2\partial_{w-2}^{[2]}\mathcal{D}_{w,2}\right) \tag{7.37}$$

Proof. Both $A_2\mathcal{D}_{w,2}$ and $\partial_{w-2}^{[2]}\mathcal{D}_{w,2}$ have weight $w+2$, depth ≤ 2 and have order $q^{\frac{3w}{4}}$, so their suitable combination has order $q^{\frac{3w}{4}+1}$. Now, we can use (7.36) to find a, b where $aA_2\mathcal{D}_{w,2} + b\partial_{w-2}^{[2]}\mathcal{D}_{w,2} = \mathcal{D}_{w+2,2}$ by comparing coefficients of $q^{\frac{3w}{4}}$ and $q^{\frac{3w}{4}+1}$, which gives the above relation. □

Corollary 7.3.17. $\mathcal{D}_{w,2}$ is completely positive if and only if $w \in \{4, 6, 8, 10, 12, 14, 18\}$.

Proof. For $w = 4$ and $w = 6$, we have the following identities:

$$\mathcal{D}'_{4,2}(z) = 3\mathcal{D}_{2,1}(z)X_{4,2}(2z) \tag{7.38}$$

$$\mathcal{D}_{6,2}(z) = \frac{1}{2880}\left(\frac{98E'_4(2z) - 13E'_4(z)}{100} + \frac{E_6(2z) - E_6(z)}{70} + A''_2(z)\right) \tag{7.39}$$

(7.38) implies that $\mathcal{D}_{4,2}$ is completely positive. From (7.39), the n-th coefficient of $\mathcal{D}_{6,2}$ is

$$\frac{1}{2880}\left[n\left(-13\sigma_3(n) + 49\sigma_3\left(\frac{n}{2}\right)\right) + 3\left(\sigma_5(n) - \sigma_5\left(\frac{n}{2}\right)\right) + 10n^2\left(\sigma_1(n) - 2\sigma_1\left(\frac{n}{2}\right)\right)\right] \tag{7.40}$$

It has order 4 at the cusp, and one can actually show that the contributions from E_4 and E_6, i.e. the σ_3 and σ_5 terms of (7.40), are already positive for $n \geq 4$.

Now, define

$$\Delta_2(z) := \eta(z)^8\eta(2z)^8 = \sum_{n\geq 1} c_n q^n = \frac{(A_2^2 - A_{4,0})(4A_{4,0} - A_2^2)}{144} \tag{7.41}$$

which is a Hecke eigenform of level $\Gamma_0(2)$ and weight 8, and its n-th coefficient c_n can be bounded by $\sigma_0(n)n^{\frac{7}{2}} < n^{\frac{9}{2}}$ by Deligne's bound. For $w \in \{8, 10, 12, 14, 18\}$, we can express $\mathcal{D}_{w,2}$ as

$$\mathcal{D}_{w,2} = P''_w + Q'_w + R_w \tag{7.42}$$

where P_w, Q_w, R_w are modular forms of level $\Gamma_0(2)$ and weights $w-4$, $w-2$, and w, respectively. Then we can decompose these into Eisenstein and cusp parts, and estimate Fourier coefficients of each part to show that the Fourier coefficients of $\mathcal{D}_{w,2} = \sum_{n\geq\lfloor\frac{3w}{4}\rfloor} d_{n,2}^{(w)}q^n$ are positive for large enough n. For example, for $w = 8$, we have

$$P_8 = \frac{E_4(z) - 8E_4(2z)}{1612800}$$
$$Q_8 = \frac{11E_6(z) - 191E_6(2z)}{71124480}$$
$$R_8 = \frac{E_8(z) - E_8(2z)}{51179520} - \frac{11\Delta_2(z)}{137088}$$

so the n-th coefficient of $\mathcal{D}_{8,2}$ is

$$d_{n,2}^{(8)} = \frac{240n^2(\sigma_3(n) - 8\sigma_3(\frac{n}{2}))}{1612800} - \frac{504n(11\sigma_5(n) - 191\sigma_5(\frac{n}{2}))}{71124480} + \frac{480(\sigma_7(n) - \sigma_7(\frac{n}{2}))}{51179520} - \frac{11c_n}{137088}$$

By using $n^k \leq \sigma_k(n) < 2n^k$ for $n \geq 1$ and $k \geq 2$, we can show that $d_{n,2}^{(8)} > 0$ for $n \geq n_0$ with effective n_0, and for $n < n_0$ we can check positivity by direct computation. For

$w = 10, 12, 14, 18$, we have the following decompositions, and we can show positivity of the Fourier coefficients similarly:

$$P_{10} = -\frac{E_6(z) + 73E_6(2z)}{731566080}$$
$$Q_{10} = -\frac{361E_8(z) + 26839E_8(2z)}{284265676800} + \frac{241\Delta_2}{84602880}$$
$$R_{10} = \frac{-E_{10}(z) + E_{10}(2z)}{2715254784} - \frac{121A_2\Delta_2}{39997440}$$
$$P_{12} = \frac{-E_8(z) + 5441E_8(2z)}{2895298560000} - \frac{\Delta_2}{5222400}$$
$$Q_{12} = \frac{151E_{10}(z) + 110953E_{10}(2z)}{53236592640000} + \frac{A_2\Delta_2}{733286400}$$
$$R_{12} = \frac{E_{12}(z) - E_{12}(2z)}{811819008000} - \frac{(333895A_2^2 - 801981A_{4,0})\Delta_2}{2451778560000}$$
$$P_{14} = \frac{-45E_{10}(z) + 18893E_{10}(2z)}{304513309900800} - \frac{A_2\Delta_2}{230461440}$$
$$Q_{14} = \frac{-22363E_{12}(z) + 20343643E_{12}(2z)}{228186736223846400} + \frac{(688459A_2^2 - 1265209A_{4,0})\Delta_2}{168290080358400}$$
$$R_{14} = \frac{-E_{14}(z) + E_{14}(2z)}{43911212826624} - \frac{A_2(8883883A_2^2 - 20201636A_{4,0})\Delta_2}{1463003388887040}$$
$$P_{18} = -\frac{5197E_{14}(z) + 2098067315E_{14}(2z)}{444164201584975872000000} - \frac{A_2(98935A_2^2 - 269494A_{4,0})\Delta_2}{21751003975680000}$$
$$Q_{18} = -\frac{581E_{16}(z) + 24329659E_{16}(2z)}{2728981492767129600000} + \frac{(61058626865A_2^4 - 197501459863A_2^2A_{4,0} + 270388013672A_{4,0}^2)\Delta_2}{1568984354155266048000000}$$
$$R_{18} = \frac{-E_{18}(z) + E_{18}(2z)}{778506238269849600} - \frac{A_2(143215428358562A_2^4 - 667208676525767A_2^2A_{4,0} + 767756329212366A_{4,0}^2)\Delta_2}{14883124757451581030400000}$$

For other (even) weights, we can show that the second nonzero coefficient of $\mathcal{D}_{w,2}$ is negative. When $w \equiv 0 \pmod 4$ and $w \geq 16$, this directly follows from (7.36). When $w \equiv 2$

(mod 4), we can use (7.36) and (7.37) to show that the second nonzero coefficient is

$$-\frac{2(w^3-18w^2-4w+8)}{(3w+2)^2} \tag{7.43}$$

which is negative for $w \geq 22$. □

Remark 7.3.18. The above decompositions of $\mathcal{D}_{w,2}$ for $w = 4, 6, 8, 10, 12, 14, 18$ are computed with the help of `ChatGPT-5.5 Pro`.

By observing the second nonzero coefficient of $\mathcal{D}_{w,2}$, we can easily show the following:

Theorem 7.3.19. There is no extremal quasimodular form of even weight, level $\Gamma_0(2)$ and depth 2 with an integral Fourier expansion.

Proof. When $w \equiv 0 \pmod 4$, assume that $\mathcal{D}_{w,2}$ has an integral Fourier expansion. Then we have

$$-\frac{2w(w^2-12w-16)}{(3w+4)^2} \in \mathbb{Z},$$

which implies

$$27(3w+4)\cdot\frac{2w(w^2-12w-16)}{(3w+4)^2} - (18w^2-240w+32) = -\frac{128}{3w+4} \in \mathbb{Z}$$

and the only possible w's are $w = 4$ or $w = 20$ (among multiples of 4). However, the corresponding (second nonzero) coefficients are $\frac{3}{2}$ and $-\frac{45}{32}$, respectively, which are not integers. The case $w \equiv 2 \pmod 4$ can be checked similarly by using (7.43). □

Higher depths

Although we do not know existence or uniqueness of extremal quasimodular forms for higher depths, one may ask whether a differential equation analogous to (7.32) is satisfied by the extremal quasimodular forms of level $\Gamma_0(2)$ and depth $s \geq 3$. Unfortunately, no such differential equation holds in general for depth ≥ 3. If $\mathcal{D}_{w,3}$ is an extremal quasimodular form of level $\Gamma_0(2)$, weight w, and depth 3, then it is natural to expect that it satisfies the following differential equation:

$$\begin{aligned}&F^{(4)} - \frac{w}{4}[E_2^{[2]}, F]_3^{(2,w-3)}\\ &= F'''' - wE_2^{[2]}F''' + \frac{3}{2}(w-1)wE_2^{[2]\prime}F'' - \frac{(w-2)(w-1)w}{2}E_2^{[2]\prime\prime}F' + \frac{(w-3)(w-2)(w-1)w}{24}E_2^{[2]\prime\prime\prime}F\end{aligned}$$

$= 0.$

However, any F satisfying the above differential equation has vanishing order w at the cusp, while the actual vanishing order of $\mathcal{D}_{w,3}$ (if it is unique) would be

$$\dim \mathcal{QM}_w^{\leq 3}(\Gamma_0(2)) - 1 = \sum_{k=0}^{3} \dim \mathcal{M}_{w-2k}(\Gamma_0(2)) - 1 = w - 1.$$

Instead, we show that the smallest-weight depth 3, level $\Gamma_0(2)$ extremal quasimodular form $\mathcal{D}_{6,3}$ is completely positive.

Proposition 7.3.20. We have

$$\mathcal{D}_{6,3}(z) = \frac{25}{84}(\mathcal{D}_{6,1}(z) - \mathcal{D}_{6,2}(z) - \mathcal{D}_{2,1}(z)X_{4,2}(2z)) \tag{7.44}$$

$$= \frac{-700E_2''(z) - 8400E_2''(2z) - 91E_4'(z) - 714E_4'(2z) - 10E_6(z) + 10E_6(2z)}{4515840} \tag{7.45}$$

$$= q^5 + \frac{10}{7}q^6 + \frac{60}{7}q^7 + \frac{80}{7}q^8 + \frac{260}{7}q^9 + \frac{324}{7}q^{10} + \cdots \tag{7.46}$$

In particular, $\mathcal{D}_{6,3}$ is completely positive.

Proof. Both identities can be checked by direct computation with Sage. By (7.45), the n-th coefficient of $\mathcal{D}_{6,3}$ is

$$a_n = \frac{A_n + B_{n/2}}{2688} \tag{7.47}$$

where

$$A_n = 10n^2\sigma_1(n) - 13n\sigma_3(n) + 3\sigma_5(n), \quad B_n = 120n^2\sigma_1(n) - 102n\sigma_3(n) - 3\sigma_5(n)$$

and $B_n = 0$ when n is not an integer. Using the elementary bounds $n^k \leq \sigma_k(n) < 2n^k$ for $k \geq 2$ (and $\sigma_1(n) < n^2$), we have

$$A_n \geq 3n^5 - 26n^4 + 10n^3, \quad B_n \geq -6n^5 - 204n^4 + 120n^3$$

and this implies $A_n > 0$ for $n \geq 9$ and $A_n + B_{n/2} > 0$ for even $n \geq 14$. Thus $a_n > 0$ for $n \geq 14$, and one can check that $a_n > 0$ for $5 \leq n \leq 13$ by direct computation. □

Remark 7.3.21. The positivity of a_n is verified in Lean 4 with the help of `Claude Opus 4.7`.

However, it is not true that all depth 3 extremal quasimodular forms of level $\Gamma_0(2)$ are completely positive. In fact, we can compute with Sage that

$$\mathcal{D}_{12,3} = q^{11} - \frac{553179}{162637}q^{12} + \frac{2138466}{162637}q^{13} - \frac{4442592}{162637}q^{14} + \cdots.$$

7.4 Level $\Gamma_0(3)$

The ring of modular forms of level $\Gamma_0(3)$ with even weights is generated by the following modular forms:

$$B_2(z) = \frac{3E_2(3z) - E_2(z)}{2} = 1 + 12q + 36q^2 + 12q^3 + 84q^4 + 72q^5 + \cdots, \tag{7.48}$$

$$B_{4,1}(z) = \frac{B_2(z)^2 - B_{4,0}(z)}{24} = q + 9q^2 + 27q^3 + 73q^4 + 126q^5 + 243q^6 + \cdots, \tag{7.49}$$

$$B_{6,2}(z) = \frac{B_2(z)^3 - 3B_2(z)B_{4,0}(z) + 2B_{6,0}(z)}{432} = q^2 + 6q^3 + 27q^4 + 80q^5 + 207q^6 + \cdots. \tag{7.50}$$

Note that $B_2, B_{4,1}, B_{6,2}$ are algebraically dependent; we have $B_{4,1}^2 = B_2 B_{6,2}$. Also, we define

$$E_2^{[3]}(z) = \frac{E_2(z) + 3B_2(z)}{4} = \frac{9E_2(3z) - E_2(z)}{8} = 1 + 3\sum_{n\geq 1}\left(\sigma_1(n) - 9\sigma_1\left(\frac{n}{3}\right)\right)q^n \tag{7.51}$$

and the corresponding *level* $\Gamma_0(3)$ *Serre derivative*

$$\partial_k^{[3]} F = F' - \frac{k}{3}E_2^{[3]}F = \partial_k F - \frac{k}{4}B_2 F. \tag{7.52}$$

Lemma 7.4.1. *$B_2, B_{4,1}, B_{6,2}$ have positive and integral Fourier coefficients.*

Proof. The positivity of B_2 is clear from

$$B_2(z) = 1 + 12\sum_{n\geq 1}\left(\sigma_1(n) - 3\sigma_1\left(\frac{n}{3}\right)\right)q^n. \tag{7.53}$$

For $B_{4,1}$, we have an analogous decomposition as (7.15):

$$\begin{aligned} B_{4,1}(z) &= \frac{B_2'(z)}{12} - \frac{E_2'(3z)}{4} + 3\left(\frac{E_2(3z) - E_2(z)}{24}\right)^2 \qquad (7.54)\\ &= \sum_{n\geq 1} n\left(\sigma_1(n) - 3\sigma_1\left(\frac{n}{3}\right)\right)q^n + 6\sum_{n\geq 1} n\sigma_1(n)q^{3n} + 3\left[\sum_{n\geq 1}\left(\sigma_1(n) - \sigma_1\left(\frac{n}{3}\right)\right)q^n\right]^2 \end{aligned}$$

which immediately shows integrality and positivity of coefficients. Finally, $B_{6,2}$ is equal to the 6-th power of

$$\frac{\eta(3z)^3}{\eta(z)} = \frac{q^{\frac{3}{8}}\prod_{n\geq 1}(1-q^{3n})^3}{q^{\frac{1}{24}}\prod_{n\geq 1}(1-q^n)} = q^{\frac{1}{3}}\sum_{n\geq 0} c_3(n)q^n, \tag{7.55}$$

where the last summation is the generating function of the number of 3-core partitions [29, 33, 4], hence has integer coefficients. □

Theorem 7.4.2. Normalized extremal modular forms of even weight w and level $\Gamma_0(3)$ are unique and have positive and integral Fourier coefficients.

Proof. From (7.7), the extremal modular forms of level $\Gamma_0(3)$ have vanishing order $\lfloor \frac{w}{3} \rfloor$ at the cusp. Hence, each of them should be of the form $B_2^a B_{4,1}^b B_{6,2}^c$ where

$$(a,b,c) = \begin{cases} (0,0,\frac{w}{6}) & w \equiv 0 \pmod 6 \\ (1,0,\frac{w-2}{6}) & w \equiv 2 \pmod 6 \\ (0,1,\frac{w-4}{6}) & w \equiv 4 \pmod 6 \end{cases} \tag{7.56}$$

so that $2a + 4b + 6c = w$ and $b + 2c = \lfloor \frac{w}{3} \rfloor$. Then the claim follows from Lemma 7.4.1. □

Remark 7.4.3. As in Remark 7.3.5, these can be used to construct a Miller basis of $\mathcal{M}_w(\Gamma_0(3))$. See Appendix A.2 for details.

Now, denote by $\mathcal{E}_w$ the unique normalized extremal quasimodular form of level $\Gamma_0(3)$, even weight w, and depth 1 (Corollary 7.2.4). As in the level $\Gamma_0(2)$ case, $\mathcal{E}_w$ for $w \equiv 0 \pmod 6$ satisfies the following differential equation:

$$\mathcal{E}_w'' - \frac{w}{3}[E_2^{[3]}, \mathcal{E}_w]_1^{(2,w-1)} = \mathcal{E}_w'' - \frac{2w}{3}E_2^{[3]}\mathcal{E}_w' + \frac{w(w-1)}{3}E_2^{[3]\prime}\mathcal{E}_w = 0. \tag{7.57}$$

Proposition 7.4.4. Write the Fourier expansion of $\mathcal{E}_w$ as

$$\mathcal{E}_w = \sum_{n \geq \lfloor \frac{2w}{3} \rfloor} e_n^{(w)} q^n = \sum_{k \geq 0} g_k(w) q^{\lfloor \frac{2w}{3} \rfloor + k} \tag{7.58}$$

with $g_k(w) = e^{(w)}_{\lfloor \frac{2w}{3} \rfloor + k}$. Assume $w \equiv 0 \pmod 6$. Then $g_k(w)$ is a rational function in w, satisfying the following identities:

$$g_k(w) = \frac{1}{\left(\frac{2w}{3} + k\right)k}\left(\sum_{i=1}^{k}\left(\frac{2w}{3}\left(\frac{2w}{3} + (k-i)\right) - \frac{w(w-1)i}{3}\right)c_i g_{k-i}(w)\right) \tag{7.59}$$

where $g_0(w) = 1$ and c_i is the i-th Fourier coefficient of $E_2^{[3]}$.

Proof. This directly follows from comparing coefficients in the differential equation (7.57). □

Here is a list of $g_k(w)$ for $1 \le k \le 4$:

$$g_1(w) = \frac{w(w+3)}{2w+3} \tag{7.60}$$
$$g_2(w) = \frac{w(w^3+45w+54)}{8(w+3/2)(w+3)} \tag{7.61}$$
$$g_3(w) = \frac{w(w+5)(w^4-14w^3+315w^2+342w-324)}{48(w+3/2)(w+3)(w+9/2)} \tag{7.62}$$
$$g_4(w) = \frac{w(w^6-27w^5+899w^4+6015w^3+11484w^2-3348w+27216)}{384(w+3/2)(w+9/2)(w+6)} \tag{7.63}$$

Theorem 7.4.5. $\mathcal{E}_w$ satisfies the following recurrence relations: for $w \equiv 0 \pmod 6$

$$\mathcal{E}_{w+2} = \frac{2w+3}{9(w+1)^2}\left(-\partial_{w-1}\mathcal{E}_w + \frac{7w+1}{12}B_2\mathcal{E}_w\right) \tag{7.64}$$
$$\mathcal{E}_{w+4} = \frac{2(w+3)}{9(w+2)^2}\left(-\partial_{w+1}\mathcal{E}_{w+2} + \frac{7w+11}{12}B_2\mathcal{E}_{w+2}\right) \tag{7.65}$$
$$\mathcal{E}_{w+6} = \frac{2(w+6)(2w+9)}{243(w+5)^2}\left(-\partial_{w+3}\mathcal{E}_{w+4} - \frac{(w+3)(w^2-8w-56)}{12(w+4)^2}B_2\mathcal{E}_{w+4} + \frac{2(w+3)(w^2+6w+7)}{3(w+4)^2}B_{4,1}\mathcal{E}_{w+2}\right) \tag{7.66}$$
$$\mathcal{E}'_{w+2} = -3(w+1)\mathcal{E}_2\mathcal{E}_{w+2} + \left(\frac{2w}{3}+1\right)B_{4,1}\mathcal{E}_w \tag{7.67}$$
$$\mathcal{E}'_{w+4} = -3(w+3)\mathcal{E}_2\mathcal{E}_{w+4} + \frac{2(w+3)}{3(w+2)^2}\left((w^2+6w+7)B_{4,1}\mathcal{E}_{w+2} - (2w+3)B_{6,2}\mathcal{E}_w\right) \tag{7.68}$$
$$\mathcal{E}'_{w+6} = -3(w+5)\mathcal{E}_2\mathcal{E}_{w+6} + \frac{4(w+3)(w+6)(2w+9)}{81(w+4)^2}\left(B_{4,1}\mathcal{E}_{w+4} - B_{6,2}\mathcal{E}_{w+2}\right) \tag{7.69}$$

Proof. Since the Serre derivative does not increase depth, the right-hand side of (7.64) is a quasimodular form of weight $w+2$ and depth ≤ 1. By uniqueness, it is enough to show that the Fourier expansion of the right-hand side starts with $q^{\lfloor \frac{2(w+2)}{3} \rfloor} = q^{\frac{2w}{3}+1}$, and this can be checked from the Fourier expansions of B_2, $\mathcal{E}_2$, and (7.60). The next two recurrence relations (7.65) and (7.66) can be checked similarly, and the remaining three relations follow from the first three. ◻

From (7.64) and (7.65), we can obtain formulas for the rational functions $g_k(w)$ when $w \equiv 2 \pmod 6$ and $w \equiv 4 \pmod 6$. For example, we have

$$g_1(w) = \begin{cases} \frac{w^2+8w-2}{2(w+1)} & w \equiv 2 \pmod 6 \\ \frac{w^2+13w+4}{2w+1} & w \equiv 4 \pmod 6 \end{cases} \tag{7.70}$$

The following corollary is an immediate consequence of the first three recurrence relations, and is analogous to Corollary 7.3.13.

Corollary 7.4.6. For even $w \geq 2$, the prime factors of the denominators of all the Fourier coefficients of $\mathcal{E}_w$ are smaller than the weight w.

As in Theorem 7.3.14, we can determine the extremal forms with integral Fourier coefficients by observing the second nonzero coefficient of $\mathcal{E}_w$.

Theorem 7.4.7. For even w, $\mathcal{E}_w$ has an integral Fourier expansion if and only if $w = 2$ or 4.

Proof. As in the proof of Theorem 7.3.14, we use (7.60) and (7.70) to find necessary conditions for the integrality of the Fourier coefficients. If $w \equiv 0 \pmod 6$, $g_1(w) = \frac{w(w+3)}{2w+3} \in \mathbb{Z}$ implies $2g_1(w) - w = \frac{3w}{2w+3} \in \mathbb{Z}$, which is impossible since $1 < \frac{3w}{2w+3} < 2$. Similarly, one can check that the only possible w's for $g_1(w) \in \mathbb{Z}$ for $w \equiv 2, 4 \pmod 6$ are $w = 2, 4, 8$. When $w = 2$ or 4, one can write

$$\mathcal{E}_2(z) = \frac{E_2(3z) - E_2(z)}{24} = \sum_{n \geq 1} \left(\sigma_1(n) - \sigma_1\left(\frac{n}{3}\right) \right) q^n,$$

$$\mathcal{E}_4(z) = \mathcal{E}_2(z)^2 + 2\left(-\frac{E_2'(3z)}{24} \right) = \left[\sum_{n \geq 1} \left(\sigma_1(n) - \sigma_1\left(\frac{n}{3}\right) \right) q^n \right]^2 + 2 \sum_{n \geq 1} n\sigma_1(n) q^{3n}$$

and this proves integrality of the Fourier coefficients. However, the coefficient of q^7 in $\mathcal{E}_8$ is $\frac{117}{7}$, which is not an integer (Table C.2). □

As in the level $\Gamma_0(2)$ case, we conjecture that $\mathcal{E}_w$ is completely positive for $w \geq 2$.

Conjecture 7.4.8. For all even $w \geq 2$, $\mathcal{E}_w$ is completely positive.

A similar approach to Proposition 7.3.8 can show that the positivity of $\mathcal{E}_w$ for $w \equiv 0 \pmod 6$ implies the positivity of $\mathcal{E}_{w+2}$, using (7.67), but this is not enough to prove the positivity of $\mathcal{E}_{w+4}$ and $\mathcal{E}_{w+6}$.

7.5 Level $\Gamma_0(4)$

The ring of quasimodular forms of level $\Gamma_0(4)$ with even weights is generated by the following modular forms:

$$C_{2,0}(z) = 2E_2(4z) - E_2(2z) = 1 + 24q^2 + 24q^4 + 96q^6 + 24q^8 + 144q^{10} + \cdots, \tag{7.71}$$

$$C_{2,1}(z) = \frac{E_2\left(z+\frac{1}{2}\right) - E_2(z)}{48} = q + 4q^3 + 6q^5 + 8q^7 + 13q^9 + 12q^{11} + \cdots. \tag{7.72}$$

Lemma 7.5.1. $C_{2,0}$ and $C_{2,1}$ have positive and integral Fourier coefficients.

Proof. The positivity of $C_{2,0}$ is clear from $C_{2,0}(z) = A_2(2z)$ and Lemma 7.3.1. For $C_{2,1}$, we can directly compute the Fourier coefficients as

$$C_{2,1}(z) = -\frac{1}{2}\sum_{n\geq 1} \sigma_1(n)((-q)^n - q^n) = \sum_{n\geq 1, 2\nmid n} \sigma_1(n) q^n$$

and the claim follows. Note that (7.16) implies $C_{2,1}(z)^2 = A_4(2z)$. □

Theorem 7.5.2. For even $w \geq 2$, the normalized extremal modular form of weight w and level $\Gamma_0(4)$ is unique and has positive and integral Fourier coefficients.

Proof. From (7.8), the extremal modular form of weight w and level $\Gamma_0(4)$ has vanishing order $\frac{w}{2}$ at the cusp. Since $C_{2,1}$ is a cusp form of weight 2, the extremal modular form is simply $C_{2,1}^{\frac{w}{2}}$, which has positive and integral Fourier coefficients. □

Let $\mathcal{F}_w(z)$ be the unique normalized extremal quasimodular form of level $\Gamma_0(4)$, weight w, and depth 1. By (7.8), we have $\dim \mathcal{QM}_w^{\leq 1}(\Gamma_0(4)) = \dim \mathcal{M}_w(\Gamma_0(4)) + \dim \mathcal{M}_{w-2}(\Gamma_0(4)) = w+1$ and the vanishing order of $\mathcal{F}_w$ at the cusp is w. Consider $\mathcal{D}_{w,1}(2z)$, which is also a quasimodular form of level $\Gamma_0(4)$, weight w, and depth 1. Since $\mathcal{D}_{w,1}(z)$ has vanishing order $\frac{w}{2}$ at the cusp, $\mathcal{D}_w(2z)$ has vanishing order w at the cusp, and hence is extremal and must coincide with $\mathcal{F}_w$ by uniqueness.

Proposition 7.5.3. For even w, $\mathcal{F}_w(z) = \mathcal{D}_{w,1}(2z)$.

As a corollary, we get positivity results for $\mathcal{F}_w$ and analogous results on denominators of Fourier coefficients and integrality of Fourier coefficients immediately.

Corollary 7.5.4. For even $w \geq 2$, the prime factors of the denominators of all the Fourier coefficients of $\mathcal{F}_w$ are smaller than w. Also, $\mathcal{F}_w$ has an integral Fourier expansion if and only if $w = 2$.

Proof. This directly follows from Proposition 7.5.3, Corollary 7.3.13, and Theorem 7.3.14. □

Table C.2 lists the first few extremal quasimodular forms of level $\Gamma_0(N)$ for $N = 2, 3, 4$ and depth 1.

Chapter 8

Feigenbaum–Grabner–Hardin's family of Fourier eigenfunctions

In [26], Feigenbaum, Grabner, and Hardin generalized the construction of magic functions in [80, 19, 14] to arbitrary dimensions d divisible by 4. Although these eigenfunctions do not prove any new optimal bounds for the sphere packing problem, it is conjectured that the corresponding quasimodular forms are positive on the imaginary axis; in fact, *complete positivity* was conjectured for the quasimodular forms of $(-1)^{d/4}$-eigenfunctions [26, Conjecture 1].

In this chapter, we prove a weak version of their conjecture for both the $(-1)^{d/4}$ and $(-1)^{d/4+1}$ families, i.e. positivity of the quasimodular forms on the imaginary axis. For the $(-1)^{d/4}$ family $\{F_w\}_w$, we prove positivity by relating these forms to extremal quasimodular forms of depth 2 (Proposition 8.1.3); the positivity then follows almost immediately from Theorem 4.4.4. For the $(-1)^{d/4+1}$ family $\{G_w\}_w$, we define a family $\{Y_w\}_w$ of "modular forms" analogous to the depth 2 extremal quasimodular forms; in particular, the vanishing order of Y_w at the cusp is one larger than that of G_w. We then prove positivity of Y_w (and hence of G_w) by showing that Y_w satisfies a hypergeometric differential equation similar to Nakaya's differential equation [57]. Note that one needs to consider $\phi_w|_S$ instead of ϕ_w in [26] to get the correct formulation (see Remark 6.4 of loc. cit.). These results will be used in Chapter 9 to give improved upper bounds for the Bourgain–Clozel–Kahane uncertainty principle in certain dimensions that are multiples of 4.

8.1 $(-1)^{d/4}$-eigenfunctions

For dimensions $d \equiv 0 \pmod 4$, the $(-1)^{d/4}$-eigenfunctions are constructed as integral transforms of quasimodular forms of level 1 and depth 2 [26, Theorem 3.2, Proposition 5.1, Proposition 5.3]. This construction recovers the (+1)-part of the magic functions of Viazovska and Cohn–Kumar–Miller–Radchenko–Viazovska for $d = 8$ and $d = 24$. To simplify computations further, we normalize the forms f_w of loc. cit. so that their first nonzero Fourier coefficients are all 1, and denote the resulting forms by F_w. The following theorem is the normalized version of their results.

Theorem 8.1.1 (Feigenbaum–Grabner–Hardin [26], normalized)**.** For even $w \geq 8$, define quasimodular forms $\{F_w\}_{w\geq 8}$ of weight w and depth 2 as

$$\begin{aligned}
F_8 &= \frac{1}{1728}(E_2^2E_4 - 2E_2E_6 + E_4^2)\\
F_{10} &= \frac{1}{1728}(-E_2^2E_6 + 2E_2E_4^2 - E_4E_6)\\
F_{12} &= \frac{1}{518400}(E_2^2E_4^2 - 2E_2E_4E_6 + E_6^2)\\
F_{14} &= \frac{1}{725760}(-E_2^2E_4E_6 + E_2E_4^3 + E_2E_6^2 - E_4^2E_6)\\
F_{16} &= \frac{1}{3657830400}(49E_2^2E_4^3 - 25E_2^2E_6^2 - 48E_2E_4^2E_6 - 25E_4^4 + 49E_4E_6^2)\\
F_{18} &= \frac{1}{2874009600}(-12E_2^2E_4^2E_6 + 5E_2E_4^4 + 19E_2E_4E_6^2 - 5E_4^3E_6 - 7E_6^3)
\end{aligned}$$

and

$$F_{w+2} = \frac{3(w-8)(w-4)}{16(w-18)(w-7)(w-6)(w-5)}\left(\frac{(w-11)(w-10)}{36}E_4F_{w-2} - \partial_{w-4}^2F_{w-2}\right) \tag{8.1}$$

$$F_{w+4} = \frac{3(w-4)w}{16(w-10)(w-5)(w-3)(w+2)}\left(\frac{(w-6)(w-5)}{36}E_4F_w - \partial_{w-2}^2F_w\right). \tag{8.2}$$

Then the vanishing order of F_w at the cusp is $\lfloor \frac{w}{4} \rfloor - 1$. For $w \equiv 0 \pmod 4$, these forms F_w satisfy the third-order ordinary differential equation

$$\partial_{w-2}^3F_w - \frac{3w^2 - 36w + 140}{144}E_4\partial_{w-2}F_w - \frac{(w-14)(w-5)(w-2)}{864}E_6F_w = 0. \tag{8.3}$$

Now, let d be a positive integer divisible by 4 and $n_+ = \lfloor (d+4)/16 \rfloor + 1$. Let $w = w_{d,+} = 12\lfloor (d+4)/16 \rfloor - d/2 + 16$. Then the following function

$$M_{d,+}(\mathbf{x}) = 4\sin^2\left(\frac{\pi\|\mathbf{x}\|^2}{2}\right)\int_0^\infty \frac{t^{2-w}F_w(i/t)}{\Delta(it)^{n_+}}e^{-\pi\|\mathbf{x}\|^2t}\mathrm{d}t \tag{8.4}$$

for $\mathbf{x} \in \mathbb{R}^d$ satisfies (here we abuse notation by writing $M_{d,+}(\mathbf{x}) = M_{d,+}(\|\mathbf{x}\|)$)

$$\widehat{M_{d,+}}(\mathbf{x}) = (-1)^{d/4} M_{d,+}(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{R}^d,$$
$$M_{d,+}(\sqrt{2n_+}) = 0 \quad \text{and} \quad M'_{d,+}(\sqrt{2n_+}) \neq 0,$$
$$M_{d,+}(\sqrt{2m}) = M'_{d,+}(\sqrt{2m}) = 0 \quad \forall m > n_+, m \in \mathbb{Z}.$$

Proof. The proof can be found in [26], except for the normalization part. When $w \equiv 0 \pmod 4$, we use the differential equation (8.3) and compare coefficients of $q^{\frac{w}{4}}$ to show that the q-expansion of F_w has the form

$$F_w = c_w \left(q^{\frac{w}{4}-1} + \frac{2(w^3 - 12w^2 + 224w - 960)}{(w-4)w} q^{\frac{w}{4}} + \cdots \right) \tag{8.5}$$

for some constant c_w, and we use induction on w with (8.2) to show that $c_w = 1$ for all $w \equiv 0 \pmod 4$. Using (8.8) (which will be proved in the next proposition), we can also show that F_w is normalized when $w \equiv 2 \pmod 4$. □

Assume that $d \equiv 0 \pmod 8$. The corresponding weight w_+ and n_+ are

$$w_+ = \begin{cases} \frac{d}{4} + 16 & d \equiv 0 \pmod{16} \\ \frac{d}{4} + 10 & d \equiv 8 \pmod{16} \end{cases}, \quad n_+ = \begin{cases} \frac{d}{16} + 1 = \frac{w}{4} - 3 & d \equiv 0 \pmod{16} \\ \frac{d-8}{16} + 1 = \frac{w}{4} - 2 & d \equiv 8 \pmod{16} \end{cases}. \tag{8.6}$$

Note that $n_+ = \lfloor \frac{d}{16} \rfloor + 1$ for any $d \equiv 0 \pmod 8$.

Positivity of F_w

In this subsection, we prove that the forms F_w are positive for all $w \geq 8$. The proof is based on Propositions 8.1.2 and 8.1.3, which reduce the positivity of F_w to the positivity of depth 2 extremal quasimodular forms.

Proposition 8.1.2. For $w \geq 12$ divisible by 4, we have

$$F_{w+2} = \frac{(w-8)(w-4)}{768(w-7)(w-5)} (E_4 F_{w-2} - E_6 F_{w-4}) \tag{8.7}$$
$$\partial_{w-2} F_w = \frac{w-5}{6} F_{w+2}. \tag{8.8}$$

Proof. The proof is similar to that of Proposition 4.4.1. We can directly check the equations for $w = 12$. For $w \geq 16$, we use induction: assume that $(8.7)_w$ and $(8.8)_w$ hold. In $(8.1)_{w+4}$, we can write $\partial_w^2 F_{w+2}$ as $\frac{6}{w-5}\partial_{w-2}^3 F_w$ using $(8.8)_w$, and then use $(8.3)_w$ (together with $(8.8)_w$ again) to express it as a combination of $E_4 F_{w+2}$ and $E_6 F_w$, proving $(8.7)_{w+4}$. Equation $(8.8)_{w+4}$ can be shown by applying ∂_{w+2} to $(8.2)_w$ and using (2.55), $(8.3)_w$, and $(8.7)_{w+4}$. □

We can now express the forms F_w in terms of depth 2 extremal quasimodular forms as follows.

Proposition 8.1.3. For $w \geq 12$ and $w \equiv 0 \pmod 4$, we have

$$F_w = -\frac{256(w-3)(w-2)(w-1)}{(w-4)w^2} X_{w,2} + E_4 X_{w-4,2} \tag{8.9}$$

$$F_{w+2} = \frac{2(w-3)}{3(w-4)} E_4 X_{w-2,2} + \frac{w-6}{3(w-4)} E_6 X_{w-4,2}. \tag{8.10}$$

Proof. Equation $(8.10)_w$ follows from $(8.9)_w$ by applying ∂_{w-2} to both sides and simplifying with (2.55), $(8.8)_{w-4}$, and $(8.8)_w$. Hence it suffices to prove (8.9). First, both F_w and $X_{w-4,2}$ are normalized (for all w), with vanishing order $\frac{w}{4} - 1$ at the cusp. Hence $E_4 X_{w-4,2} - F_w$ is a weight w, depth 2 quasimodular form with vanishing order $\frac{w}{4}$ at the cusp; that is, it is an extremal quasimodular form. By uniqueness [61], it must be a constant multiple of $X_{w,2}$: there exists a constant c_w such that $E_4 X_{w-4,2} - F_w = c_w X_{w,2}$. To compute this constant explicitly, we examine the *second* (nonzero) Fourier coefficients of F_w and $X_{w-4,2}$. Let

$$X_{w-4,2} = q^{\frac{w}{4}-1} + a_{\frac{w}{4}}^{(w-4)} q^{\frac{w}{4}} + a_{\frac{w}{4}+1}^{(w-4)} q^{\frac{w}{4}+1} + \cdots,$$
$$F_w = q^{\frac{w}{4}-1} + b_{\frac{w}{4}}^{(w)} q^{\frac{w}{4}} + b_{\frac{w}{4}+1}^{(w)} q^{\frac{w}{4}+1} + \cdots.$$

Then the Fourier expansion of $E_4 X_{w-4,2} - F_w$ is

$$\begin{aligned} E_4 X_{w-4,2} - F_w &= \left(1 + 240q + 2160q^2 + \cdots\right)\left(q^{\frac{w}{4}-1} + a_{\frac{w}{4}}^{(w-4)} q^{\frac{w}{4}} + a_{\frac{w}{4}+1}^{(w-4)} q^{\frac{w}{4}+1} + \cdots\right) \\ &\quad - \left(q^{\frac{w}{4}-1} + b_{\frac{w}{4}}^{(w)} q^{\frac{w}{4}} + b_{\frac{w}{4}+1}^{(w)} q^{\frac{w}{4}+1} + \cdots\right) \\ &= \left(240 + a_{\frac{w}{4}}^{(w-4)} - b_{\frac{w}{4}}^{(w)}\right) q^{\frac{w}{4}} + \cdots \end{aligned}$$

and we have $c_w = 240 + a_{\frac{w}{4}}^{(w-4)} - b_{\frac{w}{4}}^{(w)}$.

Thus it suffices to know an explicit formula for the *second* nonzero Fourier coefficients $a_{\frac{w}{4}}^{(w-4)}$ and $b_{\frac{w}{4}}^{(w)}$. This is provided by the following lemma.

Lemma 8.1.4. For $w \geq 12$ and $w \equiv 0 \pmod 4$, we have

$$a^{(w-4)}_{\frac{w}{4}} = \frac{2(w-4)(w^2+4w-48)}{w^2} \tag{8.11}$$

$$b^{(w)}_{\frac{w}{4}} = \frac{2(w^3-12w^2+224w-960)}{w(w-4)} \tag{8.12}$$

Proof. We will only prove (8.11); the proof of (8.12) is similar (using (8.8)). Following the computation in the proof of Proposition 3.3.4, applying ∂^2_{w-6} to $(4.31)_{w-4}$ shows that the $\frac{w}{4}$-th Fourier coefficient of $X_{w,2}$ is

$$\frac{3w^2}{16(w-1)(w-2)^2(w-3)}\left(-\frac{2w-1}{6}a^{(w-4)}_{\frac{w}{4}} + \frac{18w^2-129w+240}{3}\right)$$

(and the $(\frac{w}{4}-1)$-th Fourier coefficient vanishes). Since the forms $X_{w,2}$ are all normalized, the above expression must equal 1, and solving for $a^{(w-4)}_{\frac{w}{4}}$ gives $(8.11)_w$. □

By Lemma 8.1.4,

$$\begin{aligned} c_w &= 240 + a^{(w-4)}_{\frac{w}{4}} - b^{(w)}_{\frac{w}{4}} \\ &= \frac{240w^2(w-4) + 2(w-4)^2(w^2+4w-48) - 2w(w^3-12w^2+224w-960)}{(w-4)w^2} \\ &= \frac{256(w-3)(w-2)(w-1)}{(w-4)w^2} \end{aligned}$$

and this completes the proof of $(8.9)_w$. □

Now we can prove the positivity of F_w for all w, using these identities and the weak version of the Kaneko–Koike conjecture (Theorem 4.4.4).

Theorem 8.1.5. For all $w \geq 8$, $F_w \in \mathcal{QM}^{2,+}_w$.

Proof. By Proposition 3.3.1 and $(8.8)_w$, it is enough to show positivity of F_{w+2} for $w \equiv 0 \pmod 4$ and $w \geq 8$. Assume that F_{w-4} and F_{w-2} are positive. Since E_4 is (completely) positive and $E_6(it) \leq 0$ (resp. $E_6(it) > 0$) for $0 < t \leq 1$ (resp. $t > 1$), equation $(8.7)_w$ (resp. equation $(8.10)_w$ together with Theorem 4.4.4) implies that $F_{w+2}(it) > 0$ for $0 < t \leq 1$ (resp. $t > 1$). Hence F_{w+2} is positive. □

Remark 8.1.6. We also have the following relation between F_{w+2}, $X_{w+2,2}$, and $X_{w-2,2}$, similar to (8.9), for $w \equiv 0 \pmod 4$:

$$F_{w+2} = -\frac{256(w-6)(w-1)(w+1)}{(w-4)w^2} X_{w+2,2} + E_4 X_{w-2,2}, \tag{8.13}$$

which can be proved similarly.

As an immediate consequence of Theorem 8.1.5 and (8.4), we obtain the following nonnegativity result for the Fourier eigenfunctions $M_{d,+}$.

Corollary 8.1.7. For all d divisible by 8, we have $M_{d,+}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^d$ with $\|\mathbf{x}\| \geq \sqrt{2n_+}$, where n_+ is defined in (8.6).

Differential identities for F_w and $X_{w,2}$

In Chapter 5, we used the differential equations (5.24) and (5.25) to prove the monotonicity of F/G (Proposition 5.3.1), and similarly for dimension 24 (equations (5.36), (5.37) and Proposition 5.4.4). The equations (5.24) and (5.36) relate F_{12} and F_{16} to $X_{4,2}$ and $X_{8,2}$, respectively. The following identity generalizes these differential equations to all F_w. Let

$$L_{2,k} := \partial_k^2 - \frac{k(k+2)}{144} E_4 \tag{8.14}$$

be the modular linear differential operator of type $(k, k+4)$ [56], originating from Kaneko and Zagier [43].

Proposition 8.1.8. For all even $w \geq 12$, we have

$$L_{2,w-2} F_w = \left(\left\lfloor \frac{w}{4} \right\rfloor - 1 \right) \left(\left\lfloor \frac{w}{4} \right\rfloor - \frac{w+5}{6} \right) \Delta X_{w-8,2}. \tag{8.15}$$

In particular, $L_{2,w-2}F_w$ is positive.

Proof. Assume $4|w$. $(8.15)_w$ reduces to

$$L_{2,w-2} F_w = \frac{(w-10)(w-4)}{48} \Delta X_{w-8,2}. \tag{8.16}$$

By taking the Serre derivative of $(4.31)_{w-4}$ and using $(4.36)_{w-4}$, we get

$$\partial_{w-2} X_{w,2} = \frac{w^2}{768(w-3)(w-1)} \left(E_4 \partial_{w-6} X_{w-4,2} - \frac{w-3}{6} E_6 X_{w-4,2} \right) \tag{8.17}$$

$$\partial_{w-2}^2 X_{w,2} = \frac{w^2}{768(w-3)(w-1)} \left(E_4 \partial_{w-6}^2 X_{w-4,2} - \frac{w-1}{6} E_6 \partial_{w-6} X_{w-4,2} + \frac{w-3}{12} E_4^2 X_{w-4,2} \right). \tag{8.18}$$

Using $(8.9)_w$, we can express $L_{2,w-2}F_w$ using only extremal forms as

$$\begin{aligned} L_{2,w-2}F_w = &-\frac{256(w-1)(w-2)(w-3)}{w^2(w-4)} \partial_{w-2}^2 X_{w,2} + E_4 \partial_{w-6}^2 X_{w-4,2} \\ &+ \frac{16(w-1)(w-2)^2(w-3)}{9w(w-4)} E_4 X_{w,2} - \frac{2}{3} E_6 \partial_{w-6} X_{w-4,2} \\ &- \frac{(w+4)(w-6)}{144} E_4^2 X_{w-4,2} \end{aligned} \tag{8.19}$$

and now $(8.16)_w$ follows from $(4.31)_{w-4}$, $(8.18)_w$, $(8.19)_w$, and $(6.10)_{w-4}$ of [32], where the last equation is equivalent to

$$E_4 \partial_{w-6}^2 X_{w-4,2} + \frac{w-5}{6} E_6 \partial_{w-6} X_{w-4,2} + \frac{(w-6)^2}{144} X_{w-4,2} = \left(\frac{w-4}{4} \right)^2 \Delta X_{w-8,2}. \tag{8.20}$$

The case $w \equiv 2 \pmod 4$ can be proved by taking the Serre derivative of $(8.16)_{w-2}$ and using $(4.32)_{w-2}$, $(4.32)_{w-10}$, and $(8.3)_{w-2}$. □

Similarly, $L_{2,w-2}X_{w,2}$ is a multiple of F_{w+4}.

Proposition 8.1.9. For all even $w \geq 8$, we have

$$L_{2,w-2}X_{w,2} = \left\lfloor \frac{w}{4} \right\rfloor \left(\left\lfloor \frac{w}{4} \right\rfloor - \frac{w-1}{6} \right) F_{w+4}. \tag{8.21}$$

Proof. Use (4.31), (4.33), (8.10), and (8.13). □

8.2 $(-1)^{d/4+1}$-eigenfunctions

The construction of $(-1)^{d/4+1}$-eigenforms and eigenfunctions in [26] differs from the construction of the $(-1)^{d/4}$-eigenforms and eigenfunctions above. The corresponding "modular forms" $\{\phi_w\}_{w \geq 8}$ can be expressed in terms of the Jacobi theta functions $\Theta_2, \Theta_3, \Theta_4$, the modular discriminant function Δ, and $\log \lambda$, where λ is the modular lambda function (2.35). We find that it is more convenient to work with $\phi_w|_w S$ instead of ϕ_w, since

$\mathcal{L}_S = \log \lambda_S$ admits a Fourier expansion 2.37. Thanks to the $\mathrm{SL}_2(\mathbb{Z})$-equivariance of the Serre derivative, the forms $\phi_w|_S$ satisfy the same recurrence relations and differential equations as ϕ_w. We also find that the weight 4 and 6 modular forms G_4 and G_6 can be added to the family, and all the recurrence relations (8.22), (8.23) and the differential equation (8.24) are still satisfied at $w = 4$. The theorem below is essentially equivalent to the combination of Theorem 4.4, Proposition 5.5, and Proposition 5.6 of [26], with some minor modifications.[1]

Theorem 8.2.1 (Feigenbaum–Grabner–Hardin [26], normalized)**.** For even $w \geq 4$, define "modular forms" $\{G_w\}_{w\geq 4}$ of weight w and level $\Gamma(2)$ as

$$\begin{aligned}
G_4 &= \frac{H_2}{2^5}(H_2 + 2H_4)\\
G_6 &= \frac{H_2}{2^4}(H_2^2 + H_2H_4 + H_4^2)\\
G_8 &= \frac{H_2^3}{2^{13}}(H_2 + 2H_4)\\
G_{10} &= \frac{H_2^3}{2^{12}\cdot 5}(2H_2^2 + 5H_2H_4 + 5H_4^2)\\
G_{12} &= \frac{3\Delta\mathcal{L}_S}{2^{11}\cdot 7} + \frac{3H_2^3}{2^{20}\cdot 7}(H_2^3 + 3H_2^2H_4 + 3H_2H_4^2 + 2H_4^3)\\
G_{14} &= \frac{H_2^5}{2^{20}\cdot 7}(2H_2^2 + 7H_2H_4 + 7H_4^2)\\
G_{16} &= \frac{5E_4\Delta\mathcal{L}_S}{2^{18}\cdot 11} + \frac{5H_2^3}{2^{29}\cdot 3\cdot 11}(5H_2^5 + 20H_2^4H_4 + 42H_2^3H_4^2\\
&\qquad + 68H_2^2H_4^3 + 60H_2H_4^4 + 24H_4^5)\\
G_{18} &= -\frac{5E_6\Delta\mathcal{L}_S}{2^{17}\cdot 11\cdot 13} + \frac{5H_2^3}{2^{27}\cdot 3\cdot 11\cdot 13}(10H_2^6 + 45H_2^5H_4 + 68H_2^4H_4^2\\
&\qquad + 34H_2^3H_4^3 - 13H_2^2H_4^4 - 36H_2H_4^5 - 12H_4^6)
\end{aligned}$$

and for $w \equiv 0 \pmod 4$,

$$G_{w+2} = \frac{3(w-6)(w-2)}{16(w-16)(w-5)(w-4)(w-3)}\left(\frac{(w-9)(w-8)}{36}E_4G_{w-2} - \partial_{w-2}^2 G_{w-2}\right) \tag{8.22}$$

$$G_{w+4} = \frac{3(w-2)(w+2)}{16(w-8)(w-3)(w-1)(w+4)}\left(\frac{(w-4)(w-3)}{36}E_4G_w - \partial_w^2 G_w\right). \tag{8.23}$$

[1] There were minor errors in the expressions of ϕ_{16} and ϕ_{18} in [26], which we correct in Theorem 8.2.1.

Then the vanishing order of G_w at the cusp is $\lfloor \frac{w}{4} \rfloor - \frac{1}{2}$. For $w \equiv 0 \pmod 4$, these forms G_w satisfy the third-order ordinary differential equation

$$\partial_w^3 G_w - \frac{3w^2 - 24w + 80}{144} E_4 \partial_w G_w - \frac{(w-12)(w-3)w}{864} E_6 G_w = 0. \tag{8.24}$$

Now, let d be a positive integer divisible by 4 and $n_- = \lfloor d/16 \rfloor + 1$. Let $w = w_{d,-} = 12\lfloor d/16 \rfloor - d/2 + 14$. Then the following function

$$M_{d,-}(\mathbf{x}) = 4\sin^2\left(\frac{\pi\|\mathbf{x}\|^2}{2}\right)\int_0^\infty \frac{t^{-w} G_w(i/t)}{\Delta(it)^{n_-}} e^{-\pi\|\mathbf{x}\|^2 t} \mathrm{d}t \tag{8.25}$$

for $\mathbf{x} \in \mathbb{R}^d$ satisfies (here we abuse notation by writing $M_{d,-}(\mathbf{x}) = M_{d,-}(\|\mathbf{x}\|)$)

$$\begin{aligned} \widehat{M_{d,-}}(\mathbf{x}) &= (-1)^{d/4+1} M_{d,-}(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{R}^d, \\ M_{d,-}(\sqrt{2n_-}) &= 0 \quad \text{and} \quad M'_{d,-}(\sqrt{2n_-}) \neq 0, \\ M_{d,-}(\sqrt{2m}) &= M'_{d,-}(\sqrt{2m}) = 0 \quad \forall m > n_-, m \in \mathbb{Z}. \end{aligned}$$

Assume $d \equiv 4 \pmod 8$. Then w and n_- can be expressed as

$$w_- = \begin{cases} \frac{d}{4} + 11 & d \equiv 4 \pmod{16} \\ \frac{d}{4} + 5 & d \equiv 12 \pmod{16} \end{cases}, \quad n_- = \begin{cases} \frac{d+12}{16} = \frac{w}{4} - 2 & d \equiv 4 \pmod{16} \\ \frac{d+4}{16} = \frac{w}{4} - 1 & d \equiv 12 \pmod{16} \end{cases}. \tag{8.26}$$

Note that $n_- = \lfloor \frac{d}{16} \rfloor + 1$ for any $d \equiv 4 \pmod 8$.

To be precise, G_w is not a modular form, because of the presence of the $\mathcal{L}_S$ term. In general, it has the following form:

Proposition 8.2.2. For each w, there exists a level $\mathrm{SL}_2(\mathbb{Z})$, weight $w - 12$ modular form $\widetilde{G}_{w-12}$ and a level $\Gamma(2)$ modular form Ψ_w such that

$$G_w = \widetilde{G}_{w-12} \Delta \mathcal{L}_S + \Psi_w. \tag{8.27}$$

In particular, $\widetilde{G}_w = 0$ for $w = -8, -6, -4, -2, 2$ and $\widetilde{G}_0 = \frac{3}{2^{11} \cdot 7}$, and for $w \equiv 0 \pmod 4$ and $w \geq 12$, we have

$$\widetilde{G}_{w+4} = \frac{3(w+10)(w+14)}{16(w+4)(w+9)(w+11)(w+16)} \left(\frac{(w+8)(w+9)}{36} E_4 \widetilde{G}_w - \partial_w^2 \widetilde{G}_w\right) \tag{8.28}$$

Proof. The recurrence relations (8.22) and (8.23), together with (2.38), show by induction on w that G_w can be written in the form (8.27) (note that $\partial_{12}\Delta = 0$). To prove (8.28), it is enough to show uniqueness of (8.27); equivalently, for a modular form A of weight w and level $\mathrm{SL}_2(\mathbb{Z})$ and a modular form B of the same weight and level $\Gamma(2)$, $A\mathcal{L}_S + B = 0$ if and only if $A = B = 0$. If A and B are nonzero, then $\mathcal{L}_S = -\frac{B}{A}$ has to be a modular function of level $\Gamma(2)$, hence a rational function of the modular lambda function λ. In other words, there exists a rational function $R \in \mathbb{C}(x)$ such that

$$\mathcal{L}_S(z) = \log(1 - \lambda(z)) = R(\lambda(z)),$$

for all $z \in \mathbb{H}$, which is impossible since $\log(1 - x)$ is not a rational function in x. □

G_w admits a Fourier expansion in $q^{1/2} = e^{\pi i z}$, and these forms are normalized in the sense that the first nonzero Fourier coefficient of G_w is 1.

These forms also satisfy relations analogous to (8.7) and (8.8).

Proposition 8.2.3. For $w \geq 8$ divisible by 4, we have

$$G_{w+2} = \frac{(w-6)(w-2)}{768(w-5)(w-3)}(E_4 G_{w-2} - E_6 G_{w-4}) \tag{8.29}$$

$$\partial_w G_w = \frac{w-3}{6} G_{w+2} \tag{8.30}$$

Proof. The proof is analogous to that of Proposition 8.1.2. □

Companion of $X_{w,2}$

We can ask whether there is a family of level $\Gamma(2)$ "extremal forms" closely related to G_w, analogous to the relationship between F_w and $X_{w,2}$. We observed experimentally that the forms G_w also satisfy equations similar to (8.15); namely, $L_{2,w}G_w$ is divisible by Δ for all $w \geq 8$. This led us to the following definition of a family $\{Y_w\}_{w\geq 2}$ that satisfies the analogous equation (8.41).

Definition 8.2.4. Define Y_w inductively by

$$Y_2 = \frac{H_2}{2^4}$$

$$Y_4 = -\frac{3E_4\mathcal{L}_S}{2^{11}\cdot 5} - \frac{3}{2^{12}\cdot 5}(H_2^2 + 2H_2H_4)$$

$$Y_6 = 0$$
$$Y_8 = -\frac{5E_4^2\mathcal{L}_S}{2^{19}\cdot 7} - \frac{5}{2^{21}\cdot 3\cdot 7}(11H_2^4 + 28H_2^3H_4 + 18H_2^2H_4^2 + 12H_2H_4^3)$$

and for $w \geq 8$ with $w \equiv 0 \pmod 4$,

$$Y_{w+4} = \frac{3(w+6)^2}{16(w+3)(w+4)^2(w+5)}\left(\frac{(w+2)(w+3)}{36}E_4Y_w - \partial_w^2 Y_w\right) \tag{8.31}$$
$$Y_{w+2} = \frac{6}{w+3}\partial_w Y_w \tag{8.32}$$

The q-expansions of Y_w for $w \leq 10$ are:

$$Y_2 = \frac{H_2}{2^4} = q^{\frac{1}{2}} + 4q^{\frac{3}{2}} + 6q^{\frac{5}{2}} + 8q^{\frac{7}{2}} + 13q^{\frac{9}{2}} + \cdots$$
$$Y_4 = -\frac{3E_4\mathcal{L}_S}{2^{11}\cdot 5} - \frac{3}{2^{12}\cdot 5}(H_2^2 + 2H_2H_4)$$
$$= q^{\frac{3}{2}} + \frac{276}{25}q^{\frac{5}{2}} + \frac{1566}{35}q^{\frac{7}{2}} + \frac{14072}{105}q^{\frac{9}{2}} + \frac{113963}{385}q^{\frac{11}{2}} + \cdots$$
$$Y_6 = 0$$
$$Y_8 = -\frac{5E_4^2\mathcal{L}_S}{2^{19}\cdot 7} - \frac{5}{2^{21}\cdot 3\cdot 7}(11H_2^4 + 28H_2^3H_4 + 18H_2^2H_4^2 + 12H_2H_4^3)$$
$$= q^{\frac{5}{2}} + \frac{1020}{49}q^{\frac{7}{2}} + \frac{80470}{441}q^{\frac{9}{2}} + \frac{1593080}{1617}q^{\frac{11}{2}} + \frac{27913055}{7007}q^{\frac{13}{2}} + \cdots$$
$$Y_{10} = \frac{5E_4E_6\mathcal{L}_S}{2^{17}\cdot 7\cdot 11} - \frac{5}{2^{17}\cdot 7\cdot 11}(2H_2^5 + 11H_2^4H_4 - H_2^3H_4^2 - 24H_2^2H_4^3 - 12H_2H_4^4)$$
$$= q^{\frac{5}{2}} + \frac{2004}{49}q^{\frac{7}{2}} + \frac{259918}{441}q^{\frac{9}{2}} + \frac{84839768}{17787}q^{\frac{11}{2}} + \frac{26865297}{1001}q^{\frac{13}{2}} + \cdots.$$

Like G_w, these are *not* modular forms in general, due to the term $\mathcal{L}_S$. However, Y_w satisfies recurrence relations and differential equations analogous to those of $X_{w,2}$.

Proposition 8.2.5.

1. For $w \equiv 0 \pmod 4$, Y_w satisfies the third-order ordinary differential equation

$$\theta_w^{(2)} Y_w = \partial_w^3 Y_w - \frac{3(w+2)^2 - 4}{144}E_4\partial_w Y_w - \frac{w^2(w+3)}{864}E_6 Y_w = 0 \tag{8.33}$$

where $\theta_w^{(2)}$ is the Kaneko–Zagier operator (4.2), or equivalently,

$$Y_w''' - \frac{w+2}{4}E_2 Y_w'' + \frac{(w+1)(w+2)}{4}E_2' Y_w' - \frac{w(w+1)(w+2)}{24}E_2'' Y_w = 0. \tag{8.34}$$

2. For $w \neq 6$ and $w \equiv 0 \pmod 4$, we have

$$Y_w = q^{\frac{w+2}{4}} + \frac{2(w+2)(w^2+16w+12)}{(w+6)^2} q^{\frac{w+6}{4}} + O(q^{\frac{w+10}{4}}) \tag{8.35}$$

$$Y_{w+2} = q^{\frac{w+2}{4}} + \frac{2(w^3+30w^2+188w+72)}{(w+6)^2} q^{\frac{w+6}{4}} + O(q^{\frac{w+10}{4}}) \tag{8.36}$$

In particular, the order of Y_w at the cusp is $\lfloor \frac{w}{4} \rfloor + \frac{1}{2}$.

3. For $w \equiv 0 \pmod 4$, we have

$$Y_{w+2} = \frac{3(w+2)^2}{16(w-4)^2(w+1)(w+3)} \left(\frac{(w-3)(w-2)}{36} E_4 Y_{w-2} - \partial_{w-2}^2 Y_{w-2} \right) \tag{8.37}$$

$$Y_{w+2} = \frac{(w+2)^2}{768(w+1)(w+3)} (E_4 Y_{w-2} - E_6 Y_{w-4}) \tag{8.38}$$

$$G_w = -\frac{256(w-1)w(w+1)}{(w-2)(w+2)^2} Y_w + E_4 Y_{w-4} \tag{8.39}$$

$$G_{w+2} = \frac{2(w-1)}{3(w-2)} E_4 Y_{w-2} + \frac{w-4}{3(w-2)} E_6 Y_{w-4} \tag{8.40}$$

4. For all $w \geq 2$, Y_w can be expressed as

$$Y_w = \widetilde{Y}_w \mathcal{L}_S + \Phi_w$$

where $\widetilde{Y}_w$ (resp. Φ_w) is a holomorphic modular form of weight w and level $\mathrm{SL}_2(\mathbb{Z})$ (resp. $\Gamma(2)$).

Proof. Assume $w \equiv 0 \pmod 4$. Let R_w be the differential operator corresponding to the right-hand side of (8.31), i.e.

$$R_w = \frac{3(w+6)^2}{16(w+3)(w+4)^2(w+5)} \left(\frac{(w+2)(w+3)}{36} E_4 - \partial_w^2 \right).$$

By direct computation, one can check that $R_w(\ker \theta_w^{(2)}) \subseteq \ker \theta_{w+4}^{(2)}$; that is, if Y_w satisfies $(8.33)_w$, then $Y_{w+4} = R_w Y_w$ also satisfies $(8.33)_{w+4}$, which proves part (1). For part (2), (8.35) follows from considering the Fourier expansion of Y_w in (8.34), and (8.36) follows from (8.32) and (8.35). For part (3), (8.37) and (8.38) can be proved similarly to (8.7), and (8.39) and (8.40) can be proved similarly to (8.9) and (8.10). Finally, part (4) follows inductively from (8.31) and (8.32). □

Note that the "coefficients" of (8.31), (8.32), (8.33), (8.34), (8.37), (8.38), (8.39), (8.40) coincide with those of (4.31), (4.32), (4.36), (4.35), (4.33), (4.37), (8.9), (8.10), respectively, after replacing w with $w+2$. Identities similar to Propositions 8.1.8 and 8.1.9 hold between G_w and Y_w.

Proposition 8.2.6. For all even w, we have

$$L_{2,w}G_w = \left(\left\lfloor\frac{w}{4}\right\rfloor - \frac{1}{2}\right)\left(\left\lfloor\frac{w}{4}\right\rfloor - \frac{w+4}{6}\right)\Delta Y_{w-8} \tag{8.41}$$

$$L_{2,w}Y_w = \left(\left\lfloor\frac{w}{4}\right\rfloor + \frac{1}{2}\right)\left(\left\lfloor\frac{w}{4}\right\rfloor - \frac{w-2}{6}\right)G_{w+4} \tag{8.42}$$

Proposition 8.2.7. Y_w can be expressed as the hypergeometric series

$$Y_{4k}(z) = j(z)^{-k-\frac{1}{2}}E_4(z)^k \cdot {}_3F_2\left(\frac{4k+3}{6}, \frac{4k+5}{6}, \frac{4k+7}{6}; k+\frac{3}{2}, k+\frac{3}{2}; \frac{1728}{j(z)}\right) \tag{8.43}$$

$$Y_{4k+2}(z) = j(z)^{-k-\frac{1}{2}}E_4(z)^{k-1}E_6(z) \cdot {}_3F_2\left(\frac{4k+5}{6}, \frac{4k+7}{6}, \frac{4k+9}{6}; k+\frac{3}{2}, k+\frac{3}{2}; \frac{1728}{j(z)}\right) \tag{8.44}$$

when $z = it$ and $t \geq 1$.

Proof. The proof is similar to that of Theorem 4.4.3. □

Now, the positivity of G_w follows from the positivity of Y_w.

Theorem 8.2.8. For all even $w \geq 8$, G_w is positive, i.e. $G_w(it) > 0$ for all $t > 0$.

Proof. The proof is similar to that of Theorem 4.4.4. Positivity of $Y_w(it)$ for $t \geq 1$ follows from the hypergeometric expression in Proposition 8.2.7. Positivity for $0 < t \leq 1$ can be shown by induction using (8.37) and (8.38), as in the proof of Theorem 8.1.5. □

Corollary 8.2.9. For all $d \equiv 4 \pmod 8$, we have $M_{d,-}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^d$ with $\|\mathbf{x}\| \geq \sqrt{2n_-}$, where n_- is defined in (8.26).

Remark 8.2.10. Based on experiments, we conjecture that Y_w and G_w are completely positive, i.e. their Fourier coefficients are all nonnegative; this can be viewed as a corrected version of the conjecture suggested in [26, Remark 6.4]. It may be possible to prove that all but finitely many coefficients of each G_w are positive by using Jenkins and Pratt's bounds on Fourier coefficients of modular forms of level $\Gamma_0(2)$ [37].

Chapter 9

Improved Bounds for the Sign Uncertainty Principle

In [7], Bourgain, Clozel, and Kahane studied a variant of the uncertainty principle known as the *sign uncertainty principle*. For a suitable real-valued function f on $\mathbb{R}^d$, define $r(f)$ to be the smallest radius beyond which f is nonnegative, and define $r(\widehat{f})$ similarly for the Fourier transform. They proved that, under natural sign assumptions at the origin, the product $r(f)r(\widehat{f})$ is bounded below by a positive constant. We denote the square root of the corresponding infimum by $\mathrm{A}_+(d)$. The only currently known exact value of $\mathrm{A}_+(d)$ is $\mathrm{A}_+(12) = \sqrt{2}$, due to Cohn and Gonçalves [14]. There is also a negative version of the problem, with an analogous constant $\mathrm{A}_-(d)$; both $\mathrm{A}_+(d)$ and $\mathrm{A}_-(d)$ are known to grow on the order of $\sqrt{d}$ as $d \to \infty$.

In this chapter, we prove new upper and lower bounds in certain dimensions divisible by 4. In these dimensions, our bounds improve earlier results of Bourgain–Clozel–Kahane [7], Gonçalves–Silva–Steinerberger [30], Cohn–Gonçalves [14], and Edwin [25]. The precise statements are as follows.

Theorem 9.0.1. For multiples of 4 with $d \leq 36000$, we have

$$\mathrm{A}_+(d) \leq \sqrt{2\left\lfloor \frac{d}{16} \right\rfloor + 2}. \tag{9.1}$$

Theorem 9.0.2. For multiples of 4 with $d \le 10000$, we have

$$\mathrm{A}_{(-1)^{d/4+1}}(d) \ge \begin{cases} \sqrt{2\left\lfloor \frac{d}{24} \right\rfloor + 2} & d \not\equiv 4 \pmod{24}, \\ \sqrt{2\left\lfloor \frac{d}{24} \right\rfloor} & d \equiv 4 \pmod{24}. \end{cases} \tag{9.2}$$

Theorem 9.0.1 follows from the positivity results for the Feigenbaum–Grabner–Hardin Fourier eigenfunctions proved in the previous chapter (Theorems 8.1.5 and 8.2.8), together with computer verification that these eigenfunctions are nonpositive at the origin. Theorem 9.0.2 adapts the summation-formula method of Cohn and Gonçalves [14], using radial Schwartz summation formulae associated with *extremal Eisenstein series*.

9.1 Uncertainty principle à la Bourgain–Clozel–Kahane

Heisenberg's uncertainty principle in quantum mechanics can be expressed as an inequality between the L^2-norms of a function and its Fourier transform: for $f \in L^2(\mathbb{R})$ with $\|f\|_2 = 1$ and $\widehat{f} \in L^2(\mathbb{R})$, we have

$$\int x^2 |f(x)|^2 \mathrm{d}x \int y^2 |\widehat{f}(y)|^2 \mathrm{d}y \ge \frac{1}{16\pi^2} \tag{9.3}$$

with equality when $f(x) = e^{-\pi x^2}$. Mathematically, this means that one cannot make both f and $\widehat{f}$ too concentrated near 0. In [7], Bourgain, Clozel, and Kahane considered an alternative version of the uncertainty principle:

Question 9.1.1. For a real-valued, even, nonzero L^1 function f whose Fourier transform is also in L^1, define $r(f)$ and $r(\widehat{f})$ to be the smallest positive numbers such that $f(x) \ge 0$ (resp. $\widehat{f}(y) \ge 0$) for all $|x| \ge r(f)$ (resp. $|y| \ge r(\widehat{f})$). What is the infimum of the product $r(f)r(\widehat{f})$?

It suffices to consider Fourier $(+1)$-eigenfunctions vanishing at the origin, that is, functions satisfying $\widehat{f} = f$ and $f(0) = 0$ [7, p. 1217]. If $\mathrm{A}_+(1)$ denotes the infimum of $r(f)$ over such functions, then they proved that $\mathrm{A}_+(1)$ is positive; in fact,

$$0.4107 \cdots \approx \frac{1}{2(1+\lambda)} \le \mathrm{A}_+(1) \tag{9.4}$$

where $\lambda = -\inf_x \left(\frac{\sin x}{x}\right)$ [7, Théorème 1.1].

The same question can be formulated in higher dimensions. We say that a function $f : \mathbb{R}^d \to \mathbb{R}$ is *eventually nonnegative* if $f(\mathbf{x}) \geq 0$ for sufficiently large $\|\mathbf{x}\|$, and in that case define

$$r(f) := \inf\{R \geq 0 : f(\mathbf{x}) \geq 0 \text{ for all } \|\mathbf{x}\| \geq R\} \tag{9.5}$$

to be the radius of the last sign change of f. Let $\mathcal{A}_+(d)$ be the set of functions $f : \mathbb{R}^d \to \mathbb{R}$ such that

1. $f, \widehat{f} \in L^1(\mathbb{R}^d)$ and $\widehat{f}$ is real-valued (i.e. f is even),
2. f is eventually nonnegative while $\widehat{f}(\mathbf{0}) \leq 0$,
3. $\widehat{f}$ is eventually nonnegative while $f(\mathbf{0}) \leq 0$.

The higher-dimensional sign uncertainty principle says that

$$\mathrm{A}_+(d) := \inf_{f \in \mathcal{A}_+(d) \setminus \{0\}} \sqrt{r(f) r(\widehat{f})} > 0. \tag{9.6}$$

More precisely, Bourgain, Clozel, and Kahane proved the following lower bound.

Theorem 9.1.2 (Bourgain–Clozel–Kahane)**.** For all $d \geq 1$, we have

$$\mathrm{A}_+(d) \geq \frac{1}{\sqrt{\pi}} \left(\frac{1}{2} \Gamma\left(\frac{d}{2} + 1 \right) \right)^{\frac{1}{d}} \geq \sqrt{\frac{d}{2\pi e}}. \tag{9.7}$$

They also related the sign uncertainty principle to zeros of Dedekind zeta functions of number fields.

Proposition 9.1.3 (Bourgain–Clozel–Kahane, Proposition 4.3 of [7])**.** Suppose that there exists a number field F of degree d and discriminant D whose Dedekind zeta function $\zeta_F(s)$ has a zero in $(0, 1)$. Then

$$\mathrm{A}_+(d) \geq \sqrt{d} |D|^{-\frac{1}{2d}}. \tag{9.8}$$

In particular, if

$$\sqrt{d} |D|^{-\frac{1}{2d}} > \mathrm{A}_+(d),$$

then $\zeta_F(s)$ has no zeros in $(0, 1)$.

Using the degree-48 number field F with $\zeta_F(\frac{1}{2}) = 0$ constructed by Armitage [2], they also proved a weaker version of Theorem 9.1.2, namely that $\mathrm{A}_+(d) > 0$ for $d \equiv 0 \pmod{48}$.

9.2 Known bounds for $A_\pm(d)$

We first summarize the known bounds for $A_\pm(d)$.

Cohn–Elkies bound and $A_-(d)$

Lower bounds for $A_-(d)$ imply lower bounds for the Cohn–Elkies linear programming bound. Let Δ_d^{LP} be the infimum of $B_{r/2}(d)$ among all possible choices of $f : \mathbb{R}^d \to \mathbb{R}$ that are continuous, integrable, and satisfy the conditions in Theorem 2.4.1. Indeed, if f satisfies the conditions in Theorem 2.4.1, then $g = \widehat{f} - f$ is a nonzero (-1)-eigenfunction that is eventually nonnegative, and the last sign change of g is at most the last sign change of f. This proves

$$\Delta_d^{\mathsf{LP}} \geq B_{A_-(d)/2}(d). \tag{9.9}$$

Thus a lower bound for $A_-(d)$ gives a lower bound for Δ_d^{LP}.

Upper bounds

To prove an upper bound in a given dimension d, it suffices to construct a Fourier eigenfunction f with eigenvalue ± 1 such that $f(\mathbf{0}) = \widehat{f}(\mathbf{0}) = 0$ and $f(\mathbf{x}) \geq 0$ for $\|\mathbf{x}\| \geq r$, with $r > 0$ as small as possible. Most constructions use functions of the form

$$f(\mathbf{x}) = p(2\pi\|\mathbf{x}\|^2)e^{-\pi\|\mathbf{x}\|^2} \tag{9.10}$$

where p is a polynomial. This reduces the problem to finite-dimensional optimization, and Laguerre polynomials provide a convenient basis since (9.10) is a Fourier eigenfunction when p is a suitable linear combination of them.

For $d = 1$, Bourgain, Clozel, and Kahane proved $A_+(1) \leq 0.64$ [7, Théorème 2.1]. Later, Gonçalves, Silva, and Steinerberger improved this to 0.59355 using Hermite polynomials and numerical optimization [30, Theorem 1]; `AlphaEvolve`, Google DeepMind's evolutionary coding agent, further improved it to 0.5671 [59].

For general dimensions, Bourgain, Clozel, and Kahane used a family of functions of the form (9.10) to prove the following bound, which remains the best known uniform upper bound for $A_+(d)$.

Theorem 9.2.1 (Bourgain–Clozel–Kahane, Théorème 3.2 of [7])**.** For $d \geq 2$, we have

$$\mathrm{A}_+(d) \leq \sqrt{\frac{d+2}{2\pi}} = \left(\frac{1}{\sqrt{2\pi}} + o(1)\right)\sqrt{d} \tag{9.11}$$

where

$$\frac{1}{\sqrt{2\pi}} = 0.39894\dots. \tag{9.12}$$

Cohn and Gonçalves [14] used auxiliary functions from [18] to give an upper bound

$$\mathrm{A}_-(d) \leq (c + o(1))\sqrt{d}, \tag{9.13}$$

where c is given by

$$c = \frac{\sin(\theta/2)\cot(\theta)e^{\sec(\theta)/2}}{\sqrt{2\pi}} \approx 0.3194\dots. \tag{9.14}$$

Here $\theta \approx 1.0995\dots$ is the unique root of

$$2\log(\sec(\theta) + \tan(\theta)) = \sin(\theta) + \tan(\theta). \tag{9.15}$$

In the same paper, the authors conjectured that Bourgain–Clozel–Kahane's upper bound (9.11) cannot be improved using only functions of the form (9.10) with bounded degree of p. More precisely, let $\mathrm{A}_{\pm,N}(d)$ be the infimum of $r(f)$ over all nonzero $f : \mathbb{R}^d \to \mathbb{R}$ such that $\widehat{f} = \pm f$, $f(\mathbf{0}) = 0$, and f is of the form (9.10) with $\deg p \leq N$. Then they conjectured that [14, Conjecture 3.2]

$$\lim_{d\to\infty} \frac{\mathrm{A}_{\pm,N}(d)}{\sqrt{d}} = \frac{1}{\sqrt{2\pi}} \tag{9.16}$$

for fixed $N \geq 3$. This conjecture was proved by Cohn, Dong, and Gonçalves [12], who more generally established the same limit whenever the degree is sublinear in d.

Theorem 9.2.2 (Cohn–Dong–Gonçalves, Theorem 1.2 of [12])**.** Let $n = n(d) \geq 3$ be a sequence with $\lim_{d\to\infty} n(d)/d = 0$. Then

$$\lim_{d\to\infty} \frac{\mathrm{A}_{\pm,n(d)}(d)}{\sqrt{d}} = \frac{1}{\sqrt{2\pi}}.$$

Lower bounds

Gonçalves, Silva, and Steinerberger improved the lower bound for $A_+(1)$ to

$$A_+(1) \geq 0.45 \tag{9.17}$$

using rearrangement inequalities with an optimal-transport flavor, which do not easily generalize to higher dimensions [30, Theorem 2]. For dimensions $d \geq 2$, they proved the following lower bound instead:

Theorem 9.2.3 (Gonçalves–Silva–Steinerberger, Theorem 7 of [30])**.** For $d \geq 2$, we have

$$A_+(d) \geq \frac{1}{\sqrt{\pi}} \left(\frac{1}{1+\lambda_d} \Gamma\left(\frac{d}{2}+1\right)\right)^{\frac{1}{d}}, \tag{9.18}$$

where the number λ_d is defined in terms of the Bessel function $J_{d/2}$ as

$$\lambda_d := -\inf_{u>0} \frac{\Gamma\left(\frac{d}{2}+1\right) J_{d/2}(u)}{(u/2)^{d/2}}. \tag{9.19}$$

Moreover, $\lambda_d < \frac{1}{2}$ and $\lambda_d \to 0$ as $d \to \infty$ exponentially fast.

The two lower bounds in (9.7) and (9.18) are asymptotically the same: after division by $\sqrt{d}$, both converge to $1/\sqrt{2\pi e}$. The existence of extremizers in all dimensions was also established [30, Theorem 3].

For $A_-(d)$, Cohn and Gonçalves proved the same lower bound as Theorem 9.1.2, i.e.

$$A_-(d) > \sqrt{\frac{d}{2\pi e}}$$

for all $d \geq 1$ [14, Equation (3.2)].

Recently, Edwin [25] proved an improved lower bound for both $A_+(d)$ and $A_-(d)$ when $d \geq 5$.

Theorem 9.2.4 (Edwin, Theorem 1.4 of [25])**.** For $d \geq 5$, we have

$$A_\pm(d) \geq \sqrt{\frac{d}{4\pi}}. \tag{9.20}$$

The proof is based on the following inequality relating $\|f\|_2$, $\|f\|_1$, and $\|\widehat{f}\|_1$ [25, Theorem 1.5]: if f and $\widehat{f}$ are both in $L^1(\mathbb{R}^d)$, then

$$\|f\|_2^2 \le \left(\frac{2}{e}\right)^{\frac{d}{2}} \|f\|_1 \|\widehat{f}\|_1. \tag{9.21}$$

In particular, Edwin proved that (9.21) implies

$$\Delta_d^{\mathsf{LP}} \ge \frac{1}{4}\left(\frac{e}{8}\right)^d,$$

which has the same dominant growth rate as the Torquato–Stillinger conjectured lower bounds for the Cohn–Elkies linear program in [78] (see also [77]).

Optimal values

The exact values of $\mathrm{A}_\pm(d)$ are known in only a few dimensions. Using the magic functions of Viazovska [80] and Cohn–Kumar–Miller–Radchenko–Viazovska [19], one can show that $\mathrm{A}_-(8) = \sqrt{2}$ and $\mathrm{A}_-(24) = 2$ by considering the (-1)-eigenfunction components of the magic functions. The magic function (2.59) also shows that $\mathrm{A}_-(1) = 1$.

In [14], Cohn and Gonçalves studied the problem using modular forms. In particular, they were able to find the exact value of $\mathrm{A}_+(12)$:

Theorem 9.2.5 (Cohn–Gonçalves, Theorem 1.2 of [14])**.** $\mathrm{A}_+(12) = \sqrt{2}$.

The lower bound $\mathrm{A}_+(12) \ge \sqrt{2}$ follows from the summation formula associated with the Eisenstein series E_6:

Theorem 9.2.6 (Cohn–Gonçalves, Lemma 2.1 of [14])**.** For any radial Schwartz function $f : \mathbb{R}^{12} \to \mathbb{C}$,

$$f(0) - \sum_{j\ge 1} c_j f(\sqrt{2j}) = -\widehat{f}(0) + \sum_{j \ge 1} c_j \widehat{f}(\sqrt{2j}) \tag{9.22}$$

where $c_j = 504\sigma_5(j)$ are the coefficients of E_6.

The transformation law of E_6 (2.9) first gives the summation formula (9.22) for Gaussians; a density argument then extends it to all radial Schwartz functions. More precisely, consider the Gaussian function $f(\mathbf{x}) = e^{-\pi t|\mathbf{x}|^2}$ on $\mathbb{R}^{12}$, whose Fourier transform is

$\widehat{f}(\mathbf{y}) = t^{-6}e^{-\pi|\mathbf{y}|^2/t}$. The transformation law of E_6 (2.9) gives

$$E_6(it) = t^{-6}E_6\left(\frac{i}{t}\right) \Leftrightarrow 1 - 504\sum_{n\geq 1}\sigma_5(n)e^{-2\pi nt} = -t^{-6}\left(1 - 504\sum_{n\geq 1}\sigma_5(n)e^{-\frac{2\pi n}{t}}\right), \qquad (9.23)$$

which can be rewritten as

$$f(0) - 504\sum_{n\geq 1}\sigma_5(n)f(\sqrt{2n}) = -\widehat{f}(0) + 504\sum_{n\geq 1}\sigma_5(n)\widehat{f}(\sqrt{2n}) \qquad (9.24)$$

for Gaussians f. From this identity, one can show that the linear functional

$$f \mapsto f(0) - 504\sum_{n\geq 1}\sigma_5(n)f(\sqrt{2n}) + \widehat{f}(0) - 504\sum_{n\geq 1}\sigma_5(n)\widehat{f}(\sqrt{2n})$$

is a continuous linear functional on $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^{12})$ that vanishes on the compactly supported smooth radial functions, which are dense in $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^{12})$ (see [14, Lemma 2.1] for details). Thus (9.24) holds for all radial Schwartz functions on $\mathbb{R}^{12}$, and this implies the lower bound $\mathrm{A}_+(12) \geq \sqrt{2}$.

Lemma 9.2.7 (Cohn–Gonçalves, Lemma 2.2 of [14])**.** Let $f \in \mathcal{A}_+(12)$. If $r(f) \leq \sqrt{2}$ and $r(\widehat{f}) \leq \sqrt{2}$, then $f(\mathbf{x}) = \widehat{f}(\mathbf{x}) = 0$ whenever $|\mathbf{x}| = \sqrt{2n}$ with $n \geq 1$.

For $f \in \mathcal{A}_+(12)$, one may assume that f is radial by averaging over rotations. If f is Schwartz, then the conclusion follows almost immediately from (9.24): we have

$$f(0) + \widehat{f}(0) = \sum_{n\geq 1}\sigma_5(n)(f(\sqrt{2n}) + \widehat{f}(\sqrt{2n})) \qquad (9.25)$$

together with $f(0), \widehat{f}(0) \leq 0$ and $f(\sqrt{2n}), \widehat{f}(\sqrt{2n}) \geq 0$ for $n \geq 1$, which prove the claim. A standard mollification argument then extends the result to all of $\mathcal{A}_+(12)$. By Lemma 9.2.7, any $f \in \mathcal{A}_+(12)$ with $r(f)r(\widehat{f}) < 2$ must be identically zero [14, Lemma 2.3]; hence $\mathrm{A}_+(12) \geq \sqrt{2}$.

For the upper bound $\mathrm{A}_+(12) \leq \sqrt{2}$, Cohn and Gonçalves constructed a "magic function" inspired by Viazovska's construction, which can be expressed as

$$f(\mathbf{x}) = \sin^2\left(\frac{\pi\|\mathbf{x}\|^2}{2}\right)\int_0^\infty \frac{(H_2(it) + H_3(it))H_4(it)^3}{\Delta(it)}e^{-\pi\|\mathbf{x}\|^2 t}\,\mathrm{d}t \qquad (9.26)$$

for $\|\mathbf{x}\| > \sqrt{2}$, and can be analytically continued to the origin with $f(\mathbf{0}) = 0$.

Although exact values are rare, Afkhami-Jeddi, Cohn, Hartman, de Laat, and Tajdini conjectured that $A_+(d)/\sqrt{d}$ and $A_-(d)/\sqrt{d}$ both converge as $d \to \infty$, and that the two limits coincide [1].

Conjecture 9.2.8 (Afkhami-Jeddi–Cohn–Hartman–de Laat–Tajdini, [1, Conjecture 3.2 and Equation (3.5)]). We have

$$\lim_{d\to\infty} \frac{A_+(d)}{\sqrt{d}} = \lim_{d\to\infty} \frac{A_-(d)}{\sqrt{d}} = \frac{1}{\pi} \approx 0.3183\ldots. \tag{9.27}$$

9.3 New upper bounds for $A_+(d)$

In this section, we prove Theorem 9.0.1. The asymptotic constant in our upper bound is smaller than the constant in Bourgain–Clozel–Kahane's bound (9.11):

$$0.35355\cdots = \frac{1}{\sqrt{8}} < \frac{1}{\sqrt{2\pi}} = 0.39894\ldots.$$

In fact, the upper bound (9.1) is smaller than (9.11) for all $d \geq 52$.

The proof uses the Fourier eigenfunctions constructed in Chapter 8. More precisely, we showed that $M_{d,\pm}(\mathbf{x}) \geq 0$ for all $\|\mathbf{x}\| \geq \sqrt{2n_\pm}$, as a corollary of the positivity of the forms F_w and G_w (Corollaries 8.1.7 and 8.2.9). Hence, it suffices to show that $M_{d,\pm}(\mathbf{0}) \leq 0$, which would imply $A_\pm(d) \leq \sqrt{2n_\pm}$ for the corresponding dimensions by taking $f = M_{d,\pm}$.

Nonpositivity of $M_{d,+}(\mathbf{0})$

Write $F_w = A_w + E_2 B_{w-2} + E_2^2 C_{w-4}$ for modular forms A_w, B_{w-2}, and C_{w-4} of weights w, $w-2$, and $w-4$, respectively. Since $w \equiv 0 \pmod 4$,

$$\begin{aligned}
F_w\left(\frac{i}{t}\right) &= A_w\left(\frac{i}{t}\right) + E_2\left(\frac{i}{t}\right)B_{w-2}\left(\frac{i}{t}\right) + E_2^2\left(\frac{i}{t}\right)C_{w-4}\left(\frac{i}{t}\right) \\
&= t^w A_w(it) - \left(-t^2E_2(it) + \frac{6t}{\pi}\right)t^{w-2}B_{w-2}(it) \\
&\quad + \left(t^4E_2^2(it) - \frac{12t^3}{\pi}E_2(it) + \frac{36t^2}{\pi^2}\right)t^{w-4}C_{w-4}(it) \\
&= t^wF_w(it) - \frac{6t^{w-1}}{\pi}(B_{w-2}(it) + 2E_2(it)C_{w-4}(it)) + \frac{36t^{w-2}}{\pi^2}C_{w-4}(it)
\end{aligned}$$

Therefore the integrand of (8.4) can be expressed as

$$\frac{t^{2-w}F_w(i/t)}{\Delta(it)^{n_+}} = t^2\frac{F_w(it)}{\Delta(it)^{n_+}} - \frac{6t}{\pi}\frac{B_{w-2}(it)+2E_2(it)C_{w-4}(it)}{\Delta(it)^{n_+}} + \frac{36}{\pi^2}\frac{C_{w-4}(it)}{\Delta(it)^{n_+}}. \tag{9.28}$$

Write the relevant q-expansions as

$$\frac{F_w}{\Delta^{n_+}} = \sum_{n\geq \frac{w}{4}-n_+} a_{n,+}q^n, \quad \frac{B_{w-2}+2E_2C_{w-4}}{\Delta^{n_+}} = \sum_{n\geq 1-n_+} b_{n,+}q^n, \quad \frac{C_{w-4}}{\Delta^{n_+}} = \sum_{n\geq -n_+} c_{n,+}q^n, \tag{9.29}$$

where

$$\widetilde{F}_{w-2} := B_{w-2}+2E_2C_{w-4} = \sum_{n\geq 1}\tilde{a}^{(w-2)}_{n,+}q^n \tag{9.30}$$

is a cusp form of depth 1 and weight $w-2$. Since $\frac{w}{4}-n_+>0$, the first term in (9.28) gives a convergent integral for all $\mathbf{x}\in\mathbb{R}^d$. The other two terms have poles, which make the integral (8.4) diverge near the origin. As in [26, Proposition 2.1], following the method of [80, 19], we analytically continue the integral to the origin by subtracting the polar part. Let $\phi(t)$ denote the right-hand side of (9.28) and define the truncation $\widetilde{\phi}(t)$ by

$$\widetilde{\phi}(t) := \phi(t) - \left[-\frac{6t}{\pi}\sum_{1-n_+\leq k\leq 0} b_{k,+}e^{-2\pi kt} + \frac{36}{\pi^2}\sum_{-n_+\leq k\leq 0} c_{k,+}e^{-2\pi kt}\right] \tag{9.31}$$

The function $\widetilde{\phi}$ has exponential decay as $t\to\infty$, and (8.4) can be rewritten as

$$\begin{aligned}
M_{d,+}(\mathbf{x}) &= 4\sin^2\left(\frac{\pi\|\mathbf{x}\|^2}{2}\right)\Bigg[\int_0^\infty \widetilde{\phi}(t)e^{-\pi\|\mathbf{x}\|^2t}\mathrm{d}t \\
&\qquad - \sum_{1-n_+\leq k\leq 0}\frac{6b_{k,+}}{\pi}\int_0^\infty te^{-\pi\|\mathbf{x}\|^2t}e^{-2\pi kt}\mathrm{d}t + \sum_{-n_+\leq k\leq 0}\frac{36c_{k,+}}{\pi^2}\int_0^\infty e^{-\pi\|\mathbf{x}\|^2t}e^{-2\pi kt}\mathrm{d}t\Bigg] \\
&= 4\sin^2\left(\frac{\pi\|\mathbf{x}\|^2}{2}\right)\int_0^\infty \widetilde{\phi}(t)e^{-\pi\|\mathbf{x}\|^2t}\mathrm{d}t \\
&\qquad + 4\sin^2\left(\frac{\pi\|\mathbf{x}\|^2}{2}\right)\left[-\frac{6}{\pi}\sum_{1-n_+\leq k\leq 0}\frac{b_{k,+}}{\pi^2(\|\mathbf{x}\|^2+2k)^2} + \frac{36}{\pi^2}\sum_{-n_+\leq k\leq 0}\frac{c_{k,+}}{\pi(\|\mathbf{x}\|^2+2k)}\right]
\end{aligned}$$

which converges for $\|\mathbf{x}\|^2 > 2n_+$ and analytically continues to $\mathbf{x}=\mathbf{0}$. The only term contributing to the value at the origin is $b_{0,+}$, which gives

$$M_{d,+}(\mathbf{0}) = -\frac{6}{\pi}\cdot 4\cdot\frac{\pi^2}{4}\cdot\frac{b_{0,+}}{\pi^2} = -\frac{6b_{0,+}}{\pi}. \tag{9.32}$$

Equation (9.32) shows that $M_{d,+}(\mathbf{0}) \leq 0$ if and only if $b_{0,+} \geq 0$. Now, for each $k \geq 1$, define $p_k(n)$ as

$$\frac{1}{\prod_{n\geq 1}(1-q^n)^k} = \sum_{n\geq 0} p_k(n) q^n \tag{9.33}$$

where $p_1(n) = p(n)$ is the number of partitions of n, and $\Delta^{-m} = q^{-m} \sum_{n\geq 0} p_{24m}(n) q^n$. For $k = 0$, define $p_0(n) = \delta_{n,0}$. Equations (9.29) and (9.30) give

$$b_{0,+} = [q^{n_+}] \left(\sum_{n\geq 1} p_{24n_+}(n) q^n \right) \left(\sum_{n\geq 1} \tilde{a}^{(w-2)}_{n,+} q^n \right) = \sum_{j=1}^{n_+} p_{24n_+}(n_+ - j) \tilde{a}^{(w-2)}_{j,+}. \tag{9.34}$$

Since $p_k(n) > 0$ for all $k \geq 1$ and $n \geq 0$, it suffices to check positivity of $\tilde{a}^{(w-2)}_{j,+}$ for $1 \leq j \leq n_+$. We verified this by computer for all weights w corresponding to dimensions $d \equiv 0 \pmod 8$ with $d \leq 36000$, proving (9.1) in these dimensions.

Nonpositivity of $M_{d,-}(\mathbf{0})$

The analysis of $M_{d,-}(\mathbf{0})$ is parallel: we express it in terms of the Fourier coefficients of a modular form related to G_w.

$\log \lambda(z)$ admits the expansion [26, (A·18)]

$$\log \lambda(z) = \pi i z + 4 \log 2 + \sum_{k\geq 1} (-1)^k \frac{r_4(k)}{k} q^{\frac{k}{2}}$$

where $r_4(k)$ is the number of representations of k as a sum of four squares. Using this expansion, we can write $G_w(i/t)$ as

$$t^{-w} G_w\left(\frac{i}{t}\right) = t^{-w} \widetilde{G}_{w-12}\left(\frac{i}{t}\right) \Delta\left(\frac{i}{t}\right) \mathcal{L}_S\left(\frac{i}{t}\right) + t^{-w} \Psi_w\left(\frac{i}{t}\right) \tag{9.35}$$

$$= \widetilde{G}_{w-12}(it) \Delta(it) \log \lambda(it) + (\Psi_w|_w S)(it) \tag{9.36}$$

$$= \widetilde{G}_{w-12}(it) \Delta(it) \left(-\pi t + 4 \log 2 + \sum_{k\geq 1} (-1)^k \frac{r_4(k)}{k} e^{-\pi k t} \right) + (\Psi_w|_w S)(it) \tag{9.37}$$

Thus the integrand of (8.25) can be expressed as

$$\phi_-(t) := \frac{t^{-w} G_w(i/t)}{\Delta(it)^{n_-}} \tag{9.38}$$

$$= -\pi t \frac{\widetilde{G}_{w-12}(it)}{\Delta(it)^{n_- -1}} + \frac{\widetilde{G}_{w-12}(it) \Delta(it) (\log \lambda(it) + \pi t) + (\Psi_w|_w S)(it)}{\Delta(it)^{n_-}} \tag{9.39}$$

where the numerator of the second term admits a Fourier expansion in $q^{1/2}$. Write the Fourier expansions as

$$\frac{\widetilde{G}_{w-12}(z)}{\Delta(z)^{n_--1}} = \sum_{n\geq 1-n_-} b_{n,-}q^n, \quad \frac{\widetilde{G}_{w-12}(z)\Delta(z)(\log\lambda(z)-\pi iz)+(\Psi_w|_wS)(z)}{\Delta(z)^{n_-}} = \sum_{n\geq -2n_-} c_{n,-}q^{\frac{n}{2}}$$

and define the truncation $\widetilde{\phi}_-(t)$ of $\phi_-(t)$ as

$$\widetilde{\phi}_-(t) := \phi_-(t) - \left(-\pi t \sum_{1-n_-\leq k\leq 0} b_{k,-}e^{-2\pi kt} + \sum_{-2n_-\leq k\leq 0} c_{k,-}e^{-\pi kt}\right). \tag{9.40}$$

Then (8.25) can be rewritten as

$$\begin{aligned}
M_{d,-}(\mathbf{x}) &= 4\sin^2\left(\frac{\pi\|\mathbf{x}\|^2}{2}\right)\int_0^\infty \widetilde{\phi}_-(t)e^{-\pi\|\mathbf{x}\|^2 t}\mathrm{d}t \\
&\quad + 4\sin^2\left(\frac{\pi\|\mathbf{x}\|^2}{2}\right)\left(-\pi\sum_{1-n_-\leq k\leq 0} b_{k,-}\int_0^\infty te^{-\pi(\|\mathbf{x}\|^2+2k)t}\mathrm{d}t + \sum_{-2n_-\leq k\leq 0} c_{k,-}\int_0^\infty e^{-\pi(\|\mathbf{x}\|^2+k)t}\mathrm{d}t\right) \\
&= 4\sin^2\left(\frac{\pi\|\mathbf{x}\|^2}{2}\right)\int_0^\infty \widetilde{\phi}_-(t)e^{-\pi\|\mathbf{x}\|^2 t}\mathrm{d}t \\
&\quad + 4\sin^2\left(\frac{\pi\|\mathbf{x}\|^2}{2}\right)\left[-\pi\sum_{1-n_-\leq k\leq 0} b_{k,-}\cdot\frac{1}{\pi^2(\|\mathbf{x}\|^2+2k)^2} + \sum_{-2n_-\leq k\leq 0} c_{k,-}\cdot\frac{1}{\pi(\|\mathbf{x}\|^2+k)}\right]
\end{aligned}$$

which analytically continues to $\mathbf{x}=\mathbf{0}$. This gives

$$M_{d,-}(\mathbf{0}) = -\pi\cdot 4\cdot\frac{\pi^2}{4}\cdot\frac{b_{0,-}}{\pi^2} = -\pi b_{0,-}. \tag{9.41}$$

Thus, $M_{d,-}(\mathbf{0})\leq 0$ if and only if $b_{0,-}\geq 0$. Write the Fourier expansion of $\widetilde{G}_{w-12}$ as

$$\widetilde{G}_{w-12} = \sum_{n\geq 0}\tilde{a}_{n,-}^{(w-12)}q^n. \tag{9.42}$$

As in the case of $M_{d,+}(\mathbf{0})$, $b_{0,-}$ is given by

$$b_{0,-} = [q^{n_--1}]\left(\sum_{n\geq 0}p_{24(n_--1)}(n)q^n\right)\left(\sum_{n\geq 0}\tilde{a}_{n,-}^{(w-12)}q^n\right) = \sum_{j=0}^{n_--1}p_{24(n_--1)}(n_--1-j)\tilde{a}_{j,-}^{(w-12)} \tag{9.43}$$

where $p_k(n)$ is defined as in (9.33), hence $b_{0,-}\geq 0$ if $\tilde{a}_{j,-}^{(w-12)}\geq 0$ for all $0\leq j\leq n_--1$. We verified this by computer for all weights corresponding to dimensions $d\equiv 4 \pmod 8$ with $d\leq 36000$.

9.4 New lower bounds for $\mathrm{A}_{(-1)^{d/4+1}}(d)$

Recall that Cohn and Gonçalves proved $\mathrm{A}_+(12) \geq \sqrt{2}$ using the summation formula associated with the Eisenstein series E_6. Their argument applies more generally via the following criterion:

Proposition 9.4.1 (Cohn–Gonçalves, Proposition 2.4 of [14])**.** Let $s \in \{\pm\}$, $0 < \rho_0 < \rho_1 < \cdots$ with

$$\lim_{j\to\infty} \frac{\rho_{j+1}}{\rho_j} = 1,$$

and $c_j > 0$ for all $j \geq 1$. If every radial Schwartz function f on $\mathbb{R}^d$ satisfies the summation formula

$$f(\mathbf{0}) + s\widehat{f}(\mathbf{0}) = s\sum_{j\geq 0} c_j f(\rho_j) + \sum_{j\geq 0} c_j \widehat{f}(\rho_j),$$

then $\mathrm{A}_s(d) \geq \rho_0$.

The summation formula associated with the Eisenstein series E_{2k} for $k \geq 2$ gives $\mathrm{A}_{(-1)^{k-1}}(4k) \geq \sqrt{2}$, but this is far from optimal. To improve it, we use modular forms whose Fourier expansions start with 1, followed by as many zeros as possible, and then coefficients of a consistent sign. More precisely, if we have a modular form of weight w and level 1 with Fourier expansion

$$1 - s\sum_{n\geq n_0} c_n q^n$$

with $c_n \geq 0$ for all $n \geq n_0$, then Proposition 9.4.1 gives $\mathrm{A}_s(d) \geq \sqrt{2n_0}$. We call the modular form with the largest possible n_0 an *extremal Eisenstein series*; here n_0 is one larger than the dimension of the space of cusp forms

$$\ell = \ell_w = \dim \mathcal{S}_w(\mathrm{SL}_2(\mathbb{Z})) = \begin{cases} \left\lfloor \frac{w}{12} \right\rfloor & \text{if } w \not\equiv 2 \pmod{12}, \\ \left\lfloor \frac{w}{12} \right\rfloor - 1 & \text{if } w \equiv 2 \pmod{12}. \end{cases} \tag{9.44}$$

Such modular forms first appeared in the study of *extremal lattices*. Let $\Lambda \subset \mathbb{R}^d$ be an even unimodular lattice (so d is a multiple of 8), and let $\Theta_\Lambda(z)$ be the associated theta series (2.11), which is a modular form of weight $d/2$ for $\mathrm{SL}_2(\mathbb{Z})$. Recall that the Fourier expansion of Θ_Λ is given by (2.17). We say that Λ is *extremal* if $r_Q(1) = r_Q(2) = \cdots =$

$r_Q(\ell) = 0$. In other words, extremal lattices are even unimodular lattices with the largest possible minimal length. For example, the E_8 lattice and the Leech lattice are extremal in dimensions 8 and 24, respectively; their theta series are $E_4 = 1 + 480\sum_{n\geq 1}\sigma_3(n)q^n$ and $\frac{7}{12}E_4^3 + \frac{5}{12}E_6^2 = 1 + 196560q^2 + \cdots$. Nebe also constructed an extremal lattice in dimension 72 [58].

In [53], Mallows, Odlyzko, and Sloane proved that extremal lattices cannot exist in dimensions larger than 164000 by showing that the unique modular form of weight w with Fourier expansion

$$\mathcal{E}_w(z) = 1 + \sum_{n\geq \ell+1} a_n q^n \tag{9.45}$$

satisfies $a_{\ell+2} < 0$ when $w \equiv 0 \pmod 4$ is sufficiently large (Siegel proved that $a_{\ell+1} > 0$ for all $w \equiv 0 \pmod 4$ [71]). Jenkins and Rouse [38] studied these modular forms further and proved an explicit version of Deligne's bound for Fourier coefficients of cusp forms (not necessarily Hecke eigenforms).

Theorem 9.4.2 (Jenkins–Rouse, Theorem 1 of [38])**.** Let w be an even integer and $F \in \mathcal{S}_w(\mathrm{SL}_2(\mathbb{Z}))$ with $F(z) = \sum_{n\geq 1} a_n q^n$. Then

$$|a_n| \leq \sqrt{\log w}\left(11\sqrt{\sum_{m=1}^{\ell}\frac{|a_m|^2}{m^{w-1}}} + \frac{e^{18.72}(41.41)^{\frac{w}{2}}}{w^{\frac{w-1}{2}}}\left|\sum_{m=1}^{\ell} a_m e^{-7.288m}\right|\right) d(n) n^{\frac{w-1}{2}} \tag{9.46}$$

where $\ell = \ell_w$ and $d(n)$ is the number of divisors of n.

Using this, they obtained an explicit upper bound on the largest index at which the coefficients of $\mathcal{E}_w$ are negative when $w \equiv 0 \pmod 4$. The same argument applies when $w \equiv 2 \pmod 4$, yielding an explicit upper bound on the largest index at which the coefficients of $\mathcal{E}_w$ are positive. Since the original proof of [38, Theorem 2] contains some minor errors, we give a corrected version with a slightly larger constant.

Theorem 9.4.3. Let w be an even integer and $\mathcal{E}_w$ be the unique modular form of weight w with Fourier expansion (9.45). Assume that $w \geq 12$ and $w \neq 14$, so $\ell \geq 1$. If $w \equiv 0 \pmod 4$ (resp. $w \equiv 2 \pmod 4$), then $a_n > 0$ (resp. $a_n < 0$) if

$$n \geq e^{\frac{59.169}{w-2}}(\ell^3 \log w)^{\frac{1}{w-2}} \cdot 1.0242382\ell. \tag{9.47}$$

Proof. We assume $w \equiv 2 \pmod 4$; the other case is similar. Write $\mathcal{E}_w = E_w + h$ where E_w is the Eisenstein series (2.1) and $h = \sum_{n\geq 1} b_n q^n \in \mathcal{S}_w(\mathrm{SL}_2(\mathbb{Z}))$. Then

$$b_m = \frac{2w}{B_w}\sigma_{w-1}(m)$$

for $1 \leq m \leq \ell$. We estimate b_n for $n > \ell$ using Theorem 9.4.2. We have

$$\zeta(w) = \frac{(-1)^{w/2-1}(2\pi)^w B_w}{(w-1)!\cdot 2w} = \frac{(2\pi)^w B_w}{(w-1)!\cdot 2w}$$

and $\zeta(12) \geq \zeta(w) > 1$ gives

$$\frac{(2\pi)^w}{\zeta(12)(w-1)!} < \frac{2w}{B_w} < \frac{(2\pi)^w}{(w-1)!}. \tag{9.48}$$

Combining this with $\sigma_{w-1}(m) < \zeta(w-1)m^{w-1}$ gives

$$\begin{aligned}
\sqrt{\sum_{m=1}^{\ell}\frac{|b_m|^2}{m^{w-1}}} &< \frac{2w\zeta(w-1)}{B_w}\sqrt{\sum_{m=1}^{\ell} m^{w-1}} \\
&< \frac{\zeta(w-1)(2\pi)^w}{(w-1)!}\sqrt{\sum_{m=1}^{\ell} m^{w-1}} \\
&< \frac{\zeta(w-1)(2\pi)^w}{(w-1)!}\sqrt{\int_1^{\ell+1} x^{w-1}\mathrm{d}x} \\
&< \frac{\zeta(w-1)(2\pi)^w}{(w-1)!}\frac{(\ell+1)^{w/2}}{w^{1/2}}.
\end{aligned}$$

From (9.44), we have $w \leq 12\ell + 14$, and $(1+\frac{1}{\ell})^\ell < e$ gives

$$11\sqrt{\sum_{m=1}^{\ell}\frac{|b_m|^2}{m^{w-1}}} < \frac{11(2\pi)^w}{(w-1)!}\frac{\zeta(w-1)e^6\ell^{w/2}}{\sqrt{w}}\left(1+\frac{1}{\ell}\right)^7 \tag{9.49}$$

$$\begin{aligned}
&\leq \frac{11\zeta(11)e^6 2^7}{\sqrt{12}}\frac{(2\pi)^w\ell^{w/2}}{(w-1)!} \\
&< e^{12.008}\frac{(2\pi)^w\ell^{w/2}}{(w-1)!}.
\end{aligned} \tag{9.50}$$

For the second sum, the function $x \mapsto x^{w-1}e^{-7.288x}$ is increasing on $0 \leq x \leq \ell$ so

$$\sum_{m=1}^{\ell} b_m e^{-7.288m} < \frac{2w}{B_w}\sum_{m=1}^{\ell}\sigma_{w-1}(m)e^{-7.288m}$$

$$
\begin{aligned}
&< \frac{(2\pi)^w \zeta(w-1)}{(w-1)!} \sum_{m=1}^{\ell} m^{w-1} e^{-7.288m} \\
&< \frac{(2\pi)^w \zeta(w-1)}{(w-1)!} \ell^w e^{-7.288\ell}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{e^{18.72}(41.41)^{w/2}}{w^{\frac{w-1}{2}}} \sum_{m=1}^{\ell} b_m e^{-7.288m} &< \frac{e^{18.72}\zeta(11)(2\pi)^w \ell^{w/2}}{(w-1)!} \left(\frac{41.41\ell}{w}\right)^{w/2} \sqrt{w} e^{-7.288\cdot\frac{w-14}{12}} \\
&< \frac{e^{18.72+7.288\cdot\frac{7}{6}}\zeta(11)(2\pi)^w \ell^{w/2}}{(w-1)!} \left(\frac{41.41}{12e^{\frac{7.288}{6}}}\right)^{w/2} \sqrt{26\ell} \\
&< e^{28.859} \frac{(2\pi)^w}{(w-1)!} (1.0242382\ell)^{w/2} \sqrt{\ell}. \qquad (9.51)
\end{aligned}
$$

(Here $w \le 12\ell + 14 \le 26\ell$ is used.) Combining (9.50), (9.51), and $d(n) < 2\sqrt{n}$, we get

$$
|b_n| < e^{29.56} \frac{(2\pi)^w}{(w-1)!} \sqrt{\ell \log w} (1.0242382\ell)^{w/2} n^{w/2}. \qquad (9.52)
$$

Now, we have

$$
a_n = -\frac{2w}{B_w}\sigma_{w-1}(n) + b_n < \frac{(2\pi)^w n^{\frac{w}{2}}}{\zeta(12)(w-1)!} \left(-n^{\frac{w-2}{2}} + \zeta(12) e^{29.56} (\ell \log w)^{\frac{1}{2}} (1.0242382\ell)^{\frac{w}{2}}\right)
$$

and $a_n < 0$ if

$$
n > e^{\frac{59.169}{w-2}} (\ell^3 \log w)^{\frac{1}{w-2}} \cdot 1.0242382\ell.
$$

□

The bound is practical enough to check the sign pattern of the coefficients of $\mathcal{E}_w$ for moderate weights. For example, $\ell_{2000} = 166$, and the right-hand side of (9.47) is approximately 176.66 for $w = 2000$. Thus it suffices to check the signs of the coefficients of $\mathcal{E}_{2000}$ for $167 \le n \le 176$ in order to establish complete positivity.

Extremal Eisenstein series therefore give new lower bounds for $\mathrm{A}_{\pm}(d)$ via summation formulae analogous to (9.22). For example, if $d \equiv 4 \pmod 8$, $w = \frac{d}{2} \equiv 2 \pmod 4$, and $\mathcal{E}_w$ has a Fourier expansion of the form $\mathcal{E}_w = 1 + \sum_{n \ge \ell+1} a_n q^n$ with $a_n < 0$ for all $n \ge \ell + 1$, then the same argument as in the proof of Theorem 9.2.6 shows that $\mathrm{A}_+(d) \ge \sqrt{2(\ell+1)}$. Since $\ell + 1 = \dim \mathcal{M}_{d/2}(\mathrm{SL}_2(\mathbb{Z}))$ is asymptotic to $d/24$, this lower bound is asymptotic to $\sqrt{d/12}$, which improves the previous lower bounds in these dimensions. Similarly, when $w \equiv 0$

(mod 4) and $\mathcal{E}_w$ is completely positive, we obtain the lower bound $\mathrm{A}_-(d) \geq \sqrt{2(\ell+1)}$. For small w, the desired sign pattern follows from the following modular form identities, which can be verified directly in Sage.

Proposition 9.4.4.

$$\mathcal{E}_4 = E_4 = 1 + 240q + 2160q^2 + \cdots \tag{9.53}$$

$$\mathcal{E}_6 = E_6 = 1 - 504q - 16632q^2 - \cdots \tag{9.54}$$

$$\mathcal{E}_8 = E_8 = E_4^2 = 1 + 480q + 61920q^2 + \cdots \tag{9.55}$$

$$\mathcal{E}_{10} = E_{10} = E_4E_6 = 1 - 264q - 135432q^2 - \cdots \tag{9.56}$$

$$\mathcal{E}_{12} = \frac{7E_4^3 + 5E_6^2}{12} = \Theta_{\Lambda_{24}} = 1 + 196560q^2 + 16773120q^3 + \cdots \tag{9.57}$$

$$\mathcal{E}'_{12} = 393120X_{14,1} = 393120q^2 + 50319360q^3 + \cdots \tag{9.58}$$

$$\mathcal{E}_{14} = E_{14} = E_4^2E_6 = 1 - 24q - 196632q^2 - \cdots \tag{9.59}$$

$$\mathcal{E}_{16} = \frac{E_4(4E_4^3 + 5E_6^2)}{9} = 1 + 146880q^2 + 64757760q^3 + \cdots \tag{9.60}$$

$$\mathcal{E}'_{16} = \frac{68}{91}E_4\mathcal{E}'_{12} + 86169600X_{18,1} = 293760q^2 + 194273280q^3 + \cdots \tag{9.61}$$

$$\mathcal{E}_{18} = \frac{E_6(7E_4^3 + E_6^2)}{8} = 1 - 86184q^2 - 84575232q^3 + \cdots \tag{9.62}$$

$$\mathcal{E}'_{18} = -161336448X_{20,1} - 172368E_4X_{16,1} = 172368q^2 - 253725696q^3 + \cdots \tag{9.63}$$

$$\mathcal{E}_{20} = \frac{E_4^2(11E_4^3 + 25E_6^2)}{36} = 1 + 39600q^2 + 87859200q^3 + \cdots \tag{9.64}$$

$$\mathcal{E}'_{20} = \frac{55}{204}E_4\mathcal{E}'_{16} + 192192000X_{22,1} = 79200q^2 + 263577600q^3 + \cdots . \tag{9.65}$$

These identities show that

$$\mathrm{A}_{(-1)^{d/4+1}}(d) \geq \sqrt{2(\ell_{d/2} + 1)} \tag{9.66}$$

holds for all multiples of 4 with $4 \leq d \leq 40$. For example, $\ell_{20} = 1$, and the complete positivity of $\mathcal{E}_{20}$ yields $\mathrm{A}_-(40) \geq 2$.

For larger weights, we combine Theorem 9.4.3 with a finite check of the remaining coefficients for all even weights $w \leq 5000$, completing the proof of Theorem 9.0.2.

The following plots compare the new upper and lower bounds for $\mathrm{A}_\pm(d)$ with the previous bounds.

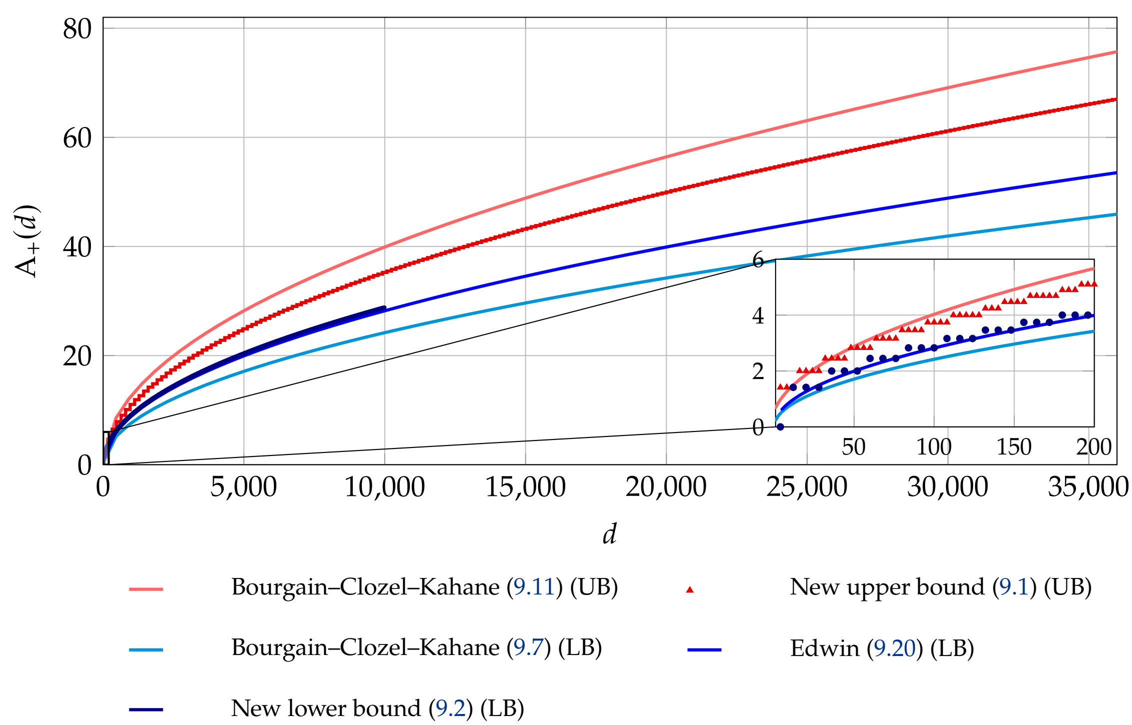


Figure 9.1: Upper and lower bounds on $A_+(d)$, with a zoomed view for $d \leq 200$ in the inset. The new bounds proved in this chapter (Theorems 9.0.1 and 9.0.2) are sampled at every multiple of 4 in the inset; the main plot shows representative samples, and the new lower bound applies in dimensions with $d \equiv 4 \pmod 8$.

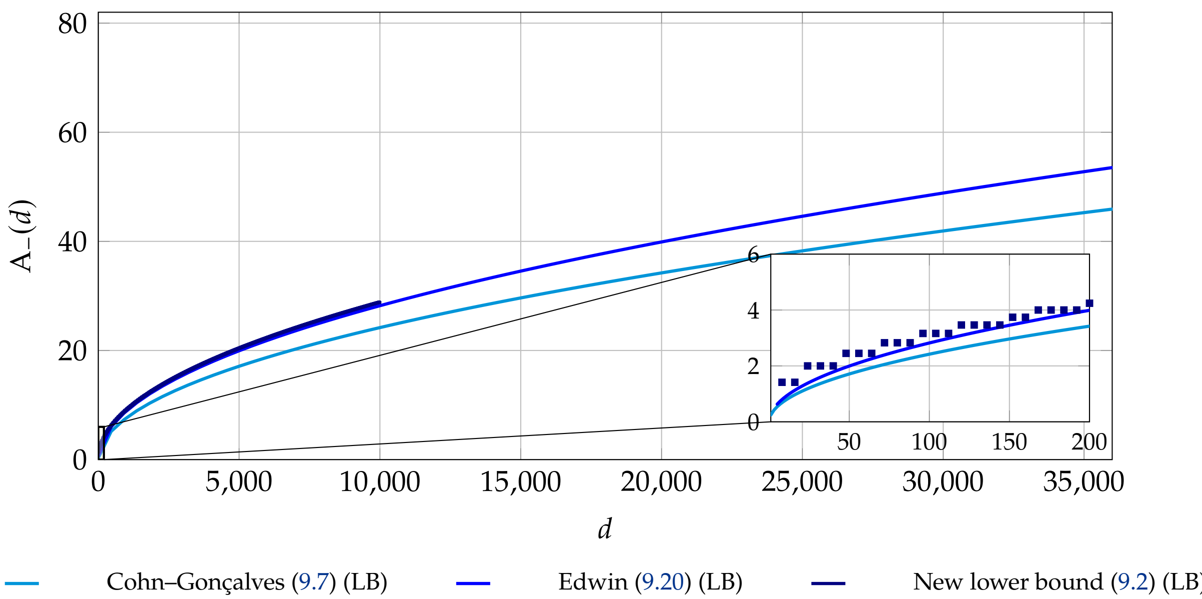


Figure 9.2: Lower bounds on $A_-(d)$, with a zoomed view for $d \leq 200$ in the inset. The new lower bound proved in this chapter (Theorem 9.0.2) applies in dimensions with $d \equiv 0 \pmod 8$.

Appendix A

Implementations in Sage

In this appendix, we give some details about the Sage implementations of the computations in this thesis. All the code is available at `https://github.com/seewoo5/posqmf`.

A.1 Quasimodular forms in Sage

Our Sage code is heavily based on the current implementation of quasimodular forms, which was largely written by David Ayotte. In Sage, the ring of quasimodular forms of level $\Gamma_0(N)$ or $\Gamma_1(N)$ is essentially implemented as a polynomial ring in one variable (E_2), with the ring of (genuine) modular forms as a coefficient ring, based on [43]. For the quasimodular forms of level $\mathrm{SL}_2(\mathbb{Z})$, we can simply define the ring as `QM = QuasiModularForms(1)`, and the Eisenstein series E_2, E_4, and E_6 are available as generators of the ring, i.e. `QM.gen(0)`, `QM.gen(1)`, and `QM.gen(2)`.

```
QM = QuasiModularForms(1)
E2, E4, E6 = QM.gen(0), QM.gen(1), QM.gen(2)
Disc = (1 / 1728) * (E4 ^ 3 - E6 ^ 2)  # discriminant form
```

A.2 Custom functionality

Extremal quasimodular forms

For given weight w and depth s, we can compute an extremal quasimodular form $X_{w,s}$ by computing a basis of the space of quasimodular forms of weight w and depth $\leq s$, and then solving a system of linear equations to determine the unique form with the desired vanishing order at infinity. However, when $s = 1$ or $s = 2$, we used Grabner's recurrence relations [32] and checked the extremality of the resulting forms by checking the vanishing order at infinity. The higher-level cases that appear in Chapter 7 are implemented essentially in the same way.

```
def is_extremal_qm(qm):
    """Check whether a quasimodular form is extremal."""
    s = qm.depth()
    w = qm.weight()
    d = dim_qm(w, s)
    order = qm_cusp_order(qm)
    return d - 1 == order

def _extremal_qm_d1(w):
    if w < 6:
        assert False, "weight should be >= 6"
    if w == 6:
        return (1 / 720) * (E2 * E4 - E6)
    else:
        if w % 6 == 0:
            _qm = _extremal_qm_d1(w - 6)
            res = E4 * qm_serre_derivative(_qm, w - 7) - ((w - 5) / 12) * E6 * _qm
            res *= w / (72 * (w - 5) * (w - 1))
            assert is_extremal_qm(res), "not extremal"
            return res
        elif w % 6 == 2:
            _qm = _extremal_qm_d1(w - 2)
            res = (12 / (w - 1)) * qm_serre_derivative(_qm, w - 3)
            assert is_extremal_qm(res), "not extremal"
            return res
```

```
        elif w % 6 == 4:
            _qm = _extremal_qm_d1(w - 4)
            res = E4 * _qm
            assert is_extremal_qm(res), "not extremal"
            return res
        else:
            assert False, "weight is odd"

def _extremal_qm_d2(w):
    if w < 4:
        assert False, "weight should be >= 4"
    if w == 4:
        return (1 / 288) * (E4 - E2^2)
    elif w % 4 == 0:
        _qm = _extremal_qm_d2(w - 4)
        res = ((w - 3) * (w - 4) / 36) * E4 * _qm
        res -= qm_serre_derivative(qm_serre_derivative(_qm, w - 6), w - 4)
        res *= (3 * (w)^2) / (16 * (w - 1) * (w - 2)^2 * (w - 3))
        assert is_extremal_qm(res), "not extremal"
        return res
    elif w % 4 == 2:
        _qm = _extremal_qm_d2(w - 2)
        res = qm_serre_derivative(_qm, w - 4)
        res *= (6 / (w - 1))
        assert is_extremal_qm(res), "not extremal"
        return res
    else:
        assert False, "weight is odd"

def extremal_qm(weight, depth):
    """
    Find the extremal quasimodular form for given weight and depth.
    The result is normalized so that its first nonzero coefficient is 1.
    """
    assert (0 <= depth <= weight/2 and 2 * (depth + 1) != weight), "inappropriate weight and
    depths"

    if depth == 1:
```

```
        return _extremal_qm_d1(weight)
    if depth == 2:
        return _extremal_qm_d2(weight)

    bs = qm_basis(weight, depth)
    d = dim_qm(weight, depth)
    m = matrix([qm_coefficients(qm_, d) for qm_ in bs])
    c_ = vector([0] * (d - 1) + [1])
    x_ = m.solve_left(c_)

    ans = sum(x_[j] * bs[j] for j in range(d))
    return ans
```

To check uniqueness of extremal forms (of general levels), it is enough to check invertibility of the $d \times d$ matrix of coefficients of the quasimodular forms in the basis of a given space, where d is the dimension of the space.

```
def is_extremal_qm_unique_level(weight, depth, level=1):
    """
    Check if the extremal quasimodular form of given weight, depth, and level is unique.
    Only supported for level Gamma0(N) for N = 1, 2, 3.
    """
    if level == 1:
        if depth <= 4:
            return True
        return is_extremal_qm_unique(weight, depth)
    elif level == 2:
        if depth <= 2:
            return True
        bs = qm_l2_basis(weight, depth)
        d = len(bs)
        m = matrix([qm_coefficients(b, d) for b in bs])
        return m.is_invertible()
    elif level == 3:
        if depth <= 1:
            return True
        bs = qm_l3_basis(weight, depth)
```

```
        d = len(bs)
        m = matrix([qm_coefficients(b, d) for b in bs])
        return m.is_invertible()
    else:
        raise NotImplementedError("Only level Gamma0(N) for N = 1, 2, 3 is supported")
```

Finding linear combinations

A large portion of the computations in this thesis involves expressing a given quasimodular form as a linear combination of other quasimodular forms, in order to study its Fourier coefficients. This is implemented as follows, by solving a system of linear equations of the first few Fourier coefficients. If the system is not solvable, then it returns an error.

```
def qm_find_lin_comb(qm, ls, prec=100):
    """Express ‘qm‘ as a linear combination of quasimodular forms in ‘ls‘."""
    w = qm.weight()
    s = qm.depth()
    m = matrix([qm_.coefficients(list(range(prec))) for qm_ in ls])
    c_ = vector(qm.coefficients(list(range(prec))))
    x_ = m.solve_left(c_)
    r = sum(x_[j] * ls[j] for j in range(len(ls)))
    assert qm == r
    return x_
```

Quasimodular forms of level $\Gamma(2)$

As mentioned before, quasimodular forms of level $\Gamma_0(N)$ or $\Gamma_1(N)$ are already available in Sage. However, for the ring of quasimodular forms of level $\Gamma(2)$, we had to implement it ourselves since the current implementation of modular forms does not support that level. The ring of quasimodular forms of level $\Gamma(2)$ is isomorphic to a polynomial ring with three generators, namely $H_2 = \Theta_2^4$, $H_4 = \Theta_4^4$, and E_2. So we simply define it as a polynomial ring `QM2.<H2,H4,E2_> = QQ[‘H2,H4,E2’]`, and implement functions that compute the q-series and (Serre) derivatives and plot the graph of a given form in t for $z = it$. We use (2.46)–(2.48) to implement (Serre) derivatives as follows.

```
QM2.<H2, H4, E2_> = QQ['H2,H4,E2']
E4_ = H2^2 + H2 * H4 + H4^2
E6_ = (H2 + 2 * H4) * (2 * H2 + H4) * (H4 - H2) / 2
Disc_ = H2^2 * (H2 + H4)^2 * H4^2 / 256

E2_2z = (1/4) * (2 * E2_ + H2 + 2 * H4)  # E2(2z)
E4_2z = (1/16) * (H2^2 + 16 * H2 * H4 + 16 * H4^2)  # E4(2z)
E6_2z = (-1/64) * (H2 + 2 * H4) * (H2^2 - 32 * H2 * H4 - 32 * H4^2)  # E6(2z)

def qm2_weight(qm):
    """Return the weight of a homogeneous level Gamma(2) quasimodular form."""
    w = None
    for (a, b, e) in qm.dict().keys():
        if w is None:
            w = 2 * a + 2 * b + 2 * e
        else:
            assert w == 2 * a + 2 * b + 2 * e
    return w

def qm2_depth(qm):
    """Return the depth of a level Gamma(2) quasimodular form."""
    dp = 0
    for (_, _, e) in qm.dict().keys():
        dp = max(e, dp)
    return dp

def qm2_derivative(qm):
    """Return the derivative of a level Gamma(2) quasimodular form."""
    r = QM2(0)
    for (a, b, e), coeff in qm.dict().items():
        r += (coeff / 6) * H2^a * H4^b * ((a - 2 * b) * H2 + (2 * a - b) * H4 + (a + b) *
    E2_) * E2_^e
        if e >= 1:
            r += coeff * H2^a * H4^b * e * E2_^(e-1) * (E2_^2 - E4_) / 12
    return r

def qm2_serre_derivative(qm, k=None):
```

```
    """Compute the level Gamma(2) Serre derivative with weight ‘k‘."""
    if k is None:
        k = qm2_weight(qm) - qm2_depth(qm)
    return qm2_derivative(qm) - (k / 12) * E2_ * qm
```

For the q-expansions of these quasimodular forms, we express them in qh $= q^{1/2}$ instead of q $= q$, since the power series ring does not support non-integer powers. We also implemented functions from the ring QM ($\mathcal{QM}(\mathrm{SL}_2(\mathbb{Z}))$) into the ring QM2 ($\mathcal{QM}(\Gamma(2))$), which correspond to the natural embedding l1_to_l2 and to $f(z) \mapsto f(2z)$ (double_argument).

```
def l1_to_l2(qm):
    r = QM2(0)
    for (d2, d4, d6), coeff in qm.polynomial.dict().items():
        r += coeff * E2_^d2 * E4_^d4 * E6_^d6
    return r

def double_argument(qm):
    """Map ‘f(z)‘ to ‘f(2z)‘ for level 1 quasimodular forms represented at level Gamma(2)."""
    r = QM2(0)
    for (d2, d4, d6), coeff in qm.polynomial().dict().items():
        r += coeff * E2_2z^d2 * E4_2z^d4 * E6_2z^d6
    return r
```

Now, we can check various quasimodular form identities using assert as follows. Note that the ring QM is implemented as a polynomial ring in three variables (namely E_2, E_4, and E_6), and Sage simply checks whether two given polynomials are equal.

```
>>> X_4_2 = extremal_qm(4, 2)
>>> X_6_1 = extremal_qm(6, 1)
>>> X_8_2 = extremal_qm(8, 2)
>>> assert X_8_2.derivative() == 2 * X_4_2 * X_6_1  # Check (51)
>>> Disc = (E4^3 - E6^2) / 1728
>>> F_8d = (E2 * E4 - E6)^2
>>> assert qm_serre_derivative_fold(F_8d, 2, 10) == (5/6) * E4 * F_8d + 172800 * Disc *
    X_4_2  # Check (64)
>>> G_24d = H2^5 * (2 * H2^2 + 7 * H2 * H4 + 7 * H4^2)
```

```
>>> assert qm2_serre_derivative_fold(G_24d, 2, 14) == (14/9) * E4_ * G_24d  # Check (72)
```

For the computations in the proof of the "harder" inequality (5.9), we define auxiliary rings `RQM` and `RQM2` corresponding to

$$\mathcal{RQM}(\Gamma) = \mathcal{QM}(\Gamma)\left[\frac{1}{\pi}, \frac{i}{z}\right]$$

for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma(2)$, by adding two formal variables `ip` and `ioz` corresponding to $1/\pi$ and $i/z = 1/t$, respectively. As mentioned before, we regard these new elements as "weight 1" objects. Then we extend the derivative D on these rings using

$$D\left(\frac{1}{\pi}\right) = 0, \quad D\left(\frac{i}{z}\right) = \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{i}{z}\right) = \frac{1}{2}\frac{1}{\pi}\left(\frac{i}{z}\right)^2$$

and the product rule.

```
# Level SL_2(Z)
RQM.<ip, ioz> = QM['ip','ioz']  # ‘ip‘ = 1 / pi, ‘ioz‘ = i / z

def rqm_weight(rqm):
    """Return the weight of a rational quasimodular form expression."""
    w = 0
    for (dip, dioz), qm in rqm.dict().items():
        w = max(w, qm.weight() + dip + dioz)
    return w

def is_rqm_homogeneous(rqm):
    """Check whether all monomials in ‘rqm‘ have the same total weight."""
    w = None
    for (dip, dioz), qm in rqm.dict().items():
        w_ = qm.weight() + dip + dioz
        if w is None:
            w = w_
        else:
            if w != w_:
                return False
    return True
```

```
def rqm_depth(rqm):
    """Return the depth of a rational quasimodular form expression."""
    dp = 0
    for qm in rqm.dict().values():
        dp = max(dp, qm_depth(qm))
    return dp

def rqm_derivative(rqm):
    """Differentiate a rational quasimodular form expression."""
    r = 0
    for (dip, dioz), qm in rqm.dict().items():
        r += qm.derivative() * ip^dip * ioz^dioz
        if dioz >= 1:
            r += qm * ip^dip * dioz * ioz^(dioz - 1) * ((1/2) * ip * ioz^2)
    return r
```

```
# Level \Gamma(2)
RQM2.<ip_, ioz_> = QM2['ip','ioz']  # ‘ip‘ = 1 / pi, ‘ioz‘ = i / z

def rqm2_weight(rqm):
    """Return the weight of a level Gamma(2) rational quasimodular form expression."""
    w = 0
    for (dip, dioz), qm in rqm.dict().items():
        w = max(w, qm2_weight(qm) + dip + dioz)
    return w

def is_rqm2_homogeneous(rqm):
    """Check whether all monomials in level Gamma(2) ‘rqm‘ share one total weight."""
    w = None
    for (dip, dioz), qm in rqm.dict().items():
        w_ = qm2_weight(qm) + dip + dioz
        if w is None:
            w = w_
        else:
```

```
        if w != w_:
            return False
    return True

def rqm2_depth(rqm):
    """Return the depth of a level Gamma(2) rational quasimodular form expression."""
    dp = 0
    for qm in rqm.dict().values():
        dp = max(dp, qm2_depth(qm))
    return dp

def rqm2_derivative(rqm):
    """Differentiate a level Gamma(2) rational quasimodular form expression."""
    r = 0
    for (dip, dioz), qm in rqm.dict().items():
        r += qm2_derivative(qm) * ip_^dip * ioz_^dioz
        if dioz >= 1:
            r += qm * ip_^dip * dioz * ioz_^(dioz - 1) * ((1/2) * ip_ * ioz_^2)
    return r
```

We can also define S-actions on these rings, using the transformation laws of Eisenstein series and theta functions. Note that, if the input is a quasimodular form without any rational terms, then the output is homogeneous. Otherwise, the output may not be homogeneous in general. Also, we always assume that the input is homogeneous. On each monomial, the action $|_w S$ on $F \cdot (1/\pi)^a \cdot (i/z)^b$ where $F \in \mathcal{QM}_{w-a-b}(\Gamma)$ equals

$$(F|_{w-a-b}S) \cdot (1/\pi)^a \cdot (i/(-1/z))^b \cdot z^{-a-b} = (-1)^{(a+b)/2} \cdot (F|_{w-a-b}S) \cdot (1/\pi)^a \cdot (i/z)^a$$

(we only deal with inputs of even weights, so $a + b$ is always even). We can extract homogeneous components using `rqm_homogeneous_comps` and `rqm2_homogeneous_comps`.

```
def rqm_S_action(rqm):
    """Apply the S-action to homogeneous rational quasimodular forms."""
    r = 0
    assert is_rqm_homogeneous(rqm), "Input is not homogeneous."
    for (dip, dioz), qm in rqm.dict().items():
        r += (-1)^((dip + dioz)/2) * qm_S_action(qm) * ip^dip * ioz^dip
```

```
    return r

def rqm2_S_action(rqm):
    """Apply the S-action to homogeneous level Gamma(2) rational forms."""
    r = 0
    assert is_rqm2_homogeneous(rqm), "Input is not homogeneous."
    for (dip, dioz), qm in rqm.dict().items():
        r += (-1)^((dip + dioz)/2) * qm2_S_action(qm) * ip_^dip * ioz_^dip
    return r

def rqm_homogeneous_comps(rqm):
    """Extract homogeneous weight components from a rational quasimodular form."""
    r = dict()
    for (dip, dioz), qm in rqm.dict().items():
        qm_comps = qm.homogeneous_components()
        for w_, qm_ in qm_comps.items():
            w = w_ + dip + dioz
            if w not in r:
                r[w] = qm_ * ip^dip * ioz^dioz
            else:
                r[w] += qm_ * ip^dip * ioz^dioz
    return r

def qm2_homogeneous_comps(qm):
    """Extract homogeneous weight components from a level Gamma(2) quasimodular form."""
    r = dict()
    for (dh2, dh4, de2), coeff in qm.dict().items():
        w = 2 * (dh2 + dh4 + de2)
        if w not in r:
            r[w] = coeff * H2^dh2 * H4^dh4 * E2_^de2
        else:
            r[w] += coeff * H2^dh2 * H4^dh4 * E2_^de2
    return r

def rqm2_homogeneous_comps(rqm):
    """Extract homogeneous weight components from level Gamma(2) rational forms."""
    r = dict()
    for (dip, dioz), qm in rqm.dict().items():
```

```
        qm_comps = qm2_homogeneous_comps(qm)
        for w_, qm_ in qm_comps.items():
            w = w_ + dip + dioz
            if w not in r:
                r[w] = qm_ * ip_^dip * ioz_^dioz
            else:
                r[w] += qm_ * ip_^dip * ioz_^dioz
    return r
```

Note that these functions also help with the limit computations such as (5.29), (5.39), and the one in Proposition 5.4.10, since we essentially use the S-action to change the limits from $\lim_{t\to 0^+}$ to $\lim_{t\to\infty}$. For example, the following code recovers the proof of Proposition 5.3.3, computing $F|_{12}S$ and $G|_{10}S$ and extracting their modular form components.

```
>>> F_8dS = qm_S_action(F_8d)
>>> G_8dS = qm2_S_action(G_8d)
>>> print_rqm(F_8dS, "F_8d|S")
F_8d|S
polynomial (E2^2*E4^2 - 2*E2*E4*E6 + E6^2) + (-12*E2*E4^2 + 12*E4*E6)*(1/π)*(i/z) +
    (36*E4^2)*(1/π)^2*(i/z)^2
weight 12
depth 2
>>> print_rqm2(G_8dS, "G_8d|S")
G_8d|S
polynomial (-5*H2^2*H4^3 - 5*H2*H4^4 - 2*H4^5)
weight 10
depth 0
>>> print("F_8dS, ip^0 * ioz^0:", F_8dS.coefficient([0, 0]))
F_8dS, ip^0 * ioz^0: 518400*q^2 + 18662400*q^3 + 255052800*q^4 + 1870387200*q^5 + O(q^6)
>>> print("F_8dS, ip^1 * ioz^1:", F_8dS.coefficient([1, 1]))
F_8dS, ip^1 * ioz^1: -8640*q - 2229120*q^2 - 56712960*q^3 - 570689280*q^4 - 3375043200*q^5 +
    O(q^6)
>>> print("F_8dS, ip^2 * ioz^2:", F_8dS.coefficient([2, 2]))
F_8dS, ip^2 * ioz^2: 36 + 17280*q + 2229120*q^2 + 37808640*q^3 + 285344640*q^4 +
    1350017280*q^5 + O(q^6)
>>> print("G_8dS, ip^0 * ioz^0:", qm2_q_series(QM2(G_8dS.coefficient([0, 0])), 10))
```

```
G_8dS, ip^0 * ioz^0: (-2) + (-240)*qh^2 + 10240*qh^3 + (-134640)*qh^4 + 1007616*qh^5 +
    (-5215680)*qh^6 + 20828160*qh^7 + (-69131760)*qh^8 + 199966720*qh^9 + Order(qh^10)
```

Victor–Miller basis

As explained in Chapter 7, Remarks 7.3.5 and 7.4.3, we can construct Victor–Miller bases of $\mathcal{M}_w(\Gamma_0(N))$ for $N = 2, 3$ using extremal modular forms. In the case of level $\Gamma_0(2)$, the modular forms $A_2^{\frac{w}{2}-i} A_{4,1}^i$ for $0 \leq i \leq \frac{w}{4}$ form a basis of $\mathcal{M}_w(\Gamma_0(2))$ of cusp orders $1, q, \ldots, q^{\frac{w}{4}}$. Then the first $d = \frac{w}{4} + 1$ coefficients of these forms form an upper-triangular matrix with diagonal entries 1, and we can invert this matrix to obtain the Victor–Miller basis.

```
def l2_victor_miller_basis(weight):
    """Return the Victor-Miller basis for level Gamma0(2) at given weight."""
    assert weight % 2 == 0, "Weight must be even"
    _M2 = ModularFormsRing(Gamma0(2))
    A2 = _M2.0  # 2E2(2z) - E2(z)
    A4_0 = _M2.1  # E4(2z)
    A4_1 = (A2 ^ 2 - A4_0) / 48
    if weight % 4 == 0:
        tri_basis = [A2 ^ (weight // 2 - 2 * i) * A4_1 ^ i for i in range(weight // 4 + 1)]
    else:  # weight % 4 == 2
        tri_basis = [A2 ^ (weight // 2 - 2 * i) * A4_1 ^ i for i in range((weight - 2) // 4 +
1)]
    d = len(tri_basis)
    T = identity_matrix(QQ, d)
    for i, f in enumerate(tri_basis):
        f_qexp = f.qexp(d).list()  # can be shorter than d
        for j in range(i + 1, len(f_qexp)):
            T[i, j] = f_qexp[j]
    Tinv = T.inverse()
    vm_basis = [sum(Tinv[i, j] * tri_basis[j] for j in range(d)) for i in range(d)]
    return vm_basis
```

Similarly, we can get the triangular basis of $\mathcal{M}_w(\Gamma_0(3))$ using the forms $B_2^i B_{4,1}^j B_{6,2}^k$ for $i, j, k \in \mathbb{Z}_{\geq 0}$ and $k \in \{0, 1\}$ such that $2i + 4j + 6k = w$. The form has cusp order $j + 2k$, and these cusp orders are all distinct for different basis elements; if $2i_1+4j_1+6k_1 = 2i_2+4j_2+6k_2$ and $j_1 + 2k_1 = j_2 + 2k_2$, then $j_1 - j_2 = -2(i_1 - i_2) = -2(k_1 - k_2)$, and $j_1, j_2 \in \{0, 1\}$, so $j_1 = j_2$ and $i_1 = i_2, k_1 = k_2$. Then we can get the Victor–Miller basis by sorting the basis elements in ascending order of their cusp orders and inverting the upper-triangular matrix of coefficients of these forms.

```
def l3_victor_miller_basis(weight):
    """Return the Victor-Miller basis for level Gamma0(3) at given weight."""
    assert weight % 2 == 0, "Weight must be even"
    _M3 = ModularFormsRing(Gamma0(3))
    B2 = _M3.0  # (3E2(3z) - E2(z)) / 2
    B4_0 = _M3.1  # E4(3z)
    B6_0 = _M3.2  # E6(3z)
    B4_1 = (B2 ^ 2 - B4_0) / 24
    B6_2 = (B2 ^ 3 - 3 * B2 * B4_0 + 2 * B6_0) / 432
    tri_basis = []
    for i in range(weight // 2 + 1):
        if (weight - 2 * i) % 6 == 0:
            k = (weight - 2 * i) // 6
            tri_basis.append(((i, 0, k), B2 ^ i * B6_2 ^ k))
        if (weight - 2 * i - 4) % 6 == 0:
            k = (weight - 2 * i - 4) // 6
            tri_basis.append(((i, 1, k), B2 ^ i * B4_1 * B6_2 ^ k))
    tri_basis.sort(key=lambda x: x[0][1] + 2 * x[0][2])
    tri_basis = [f for _, f in tri_basis]
    d = len(tri_basis)
    T = identity_matrix(QQ, d)
    for i, f in enumerate(tri_basis):
        f_qexp = f.qexp(d).list()  # can be shorter than d
        for j in range(i + 1, len(f_qexp)):
            T[i, j] = f_qexp[j]
    Tinv = T.inverse()
    vm_basis = [sum(Tinv[i, j] * tri_basis[j] for j in range(d)) for i in range(d)]
    return vm_basis
```

For example, we get the following basis elements for weight $w = 8$ and level $\Gamma_0(2)$:

$$f_0 = -\frac{5}{24}A_2^4 + \frac{5}{12}A_2^2 A_{4,0} + \frac{19}{24}A_{4,0}^2 = 1 - 7680q^3 + 4320q^4 - 276480q^5 + \cdots,$$
$$f_1 = -\frac{1}{288}A_2^4 + \frac{1}{36}A_2^2 A_{4,0} - \frac{7}{288}A_{4,0}^2 = q + 140q^3 + 1024q^4 + 4398q^5 + \cdots,$$
$$f_2 = \frac{1}{2304}A_2^4 - \frac{1}{1152}A_2^2 A_{4,0} + \frac{1}{2304}A_{4,0}^2 = q^2 + 16q^3 + 120q^4 + 576q^5 + \cdots.$$

and similarly for level $\Gamma_0(3)$:

$$f_0 = B_{4,0}^2 = 1 + 480q^3 + 61920q^6 + \cdots,$$
$$f_1 = -\frac{1}{48}B_2^2 B_{4,0} + \frac{1}{8}B_2 B_{6,0} - \frac{5}{48}B_{4,0}^2 = q - 135q^3 - 902q^4 - \cdots,$$
$$f_2 = \frac{1}{144}B_2^2 B_{4,0} - \frac{1}{72}B_2 B_{6,0} + \frac{1}{144}B_{4,0}^2 = q^2 + 18q^3 + 135q^4 + \cdots.$$

Feigenbaum–Grabner–Hardin Fourier eigenfunctions

In Chapter 9, we used Fourier eigenfunctions constructed by Feigenbaum, Grabner, and Hardin [26] to obtain new upper bounds for the sign uncertainty principle constant $\mathrm{A}_+(d)$. For this argument, we must prove the nonpositivity of $M_{d,\pm}(\mathbf{0})$, which reduces to verifying the nonnegativity of the first few Fourier coefficients of certain quasimodular forms.

For $(-1)^{d/4}$-eigenforms, we implemented F_w and $\widetilde{F}_w$ using the recursive formulas (8.1), (8.2), and (9.30). For $(-1)^{d/4+1}$-eigenforms, the $\mathcal{L}_S$ term must also be tracked. We did this by adjoining a generator for $\mathcal{L}_S$, namely `QM2_LS.<LS> = QM2['LS']`, and by implementing functions that compute derivatives, q-series, and the modular components A and B of an expression $G = A + B\mathcal{L}_S$. We then used these routines to verify the nonnegativity of the required initial Fourier coefficients of $\widetilde{G}_w$. The computation was carried out on the high-performance computing server of the UC Berkeley Department of Mathematics. It took more than a month to verify the required coefficients for all dimensions up to 36000.

Extremal Eisenstein series

Extremal Eisenstein series $\mathcal{E}_w$ are used in Chapter 9 to give new lower bounds for the sign uncertainty principle constant $\mathrm{A}_{(-1)^{d/4+1}}(d)$. In Sage, the default basis of a space of modular forms is the Victor–Miller basis for the cusp forms, followed by Eisenstein series. When the level is 1, the Eisenstein series is simply E_w of (2.2). Hence, if we denote the elements of the Victor–Miller basis as $f_1, \ldots, f_\ell$, then the extremal Eisenstein series can be computed by

$$\mathcal{E}_w = \frac{2w}{B_w} \sum_{n=1}^{\ell} \sigma_{w-1}(n) f_n + E_w$$

and we do not need to solve any linear equations for this.

```
def extremal_eisenstein_series(weight):
    """
    Compute extremal Eisenstein series of given weight, i.e. a modular form of weight $w$
    with $q$-expansion
    $1 + O(q^{l})$ where $l$ is the dimension of the space of modular forms of weight $w$.
    """
    assert weight % 2 == 0 and weight >= 4, "Weight must be even and >= 4"
    w = weight
    basis = ModularForms(weight=weight).basis()
    # The last basis element is the Eisenstein series, and the rest are Victor-Miller basis.
    d = len(basis)  # d = l + 1
    cs = []
    for n in range(1, d):
        coeff = (2 * w) / bernoulli(w) * sigma(n, w - 1)
        cs.append(coeff)
    r = sum(c * b for (c, b) in zip(cs, basis[:-1]))
    r += basis[-1]
    return r
```

Appendix B

Formalization of Sphere Packing in Dimension 8 in Lean 4

In this appendix, we briefly explain the `Sphere-Packing-Lean` project, which is an ongoing project formalizing Viazovska's proof of the optimality of the E_8-lattice in dimension 8 in Lean 4. We also show some examples of the formalized statements, which we expect to be updated as the project progresses. See also [36] for a short summary of the current status of the project.

B.1 Overview

As mentioned earlier in the introduction, the project was initiated by Sidharth Hariharan while he was visiting Viazovska at EPFL as an exchange student. Its existence was first announced at the ICMS workshop "Formalisation of Mathematics: Workshop for Women and Mathematicians of Minority Gender" in May 2024. Chris Birkbeck, Gareth Ma, Bhavik Mehta, and I then joined the project, setting up the early blueprint and formalizing the core definitions and related theorems during summer 2024. In particular, we decided to follow the proof in Chapter 5 instead of Viazovska's original proof. The project repository was made public at the "Big Proof: formalizing mathematics at scale" workshop at the Isaac Newton Institute in Cambridge in June 2025, and it began to receive numerous contributions from the community, both from humans and AI systems (and combinations of the two). Several companies working on autoformalization (i.e., automatically formalizing mathematical results using AI or other tools) also began contributing to the project, including Harmonic (with their model Aristotle) and Math Inc (with their model Gauss).

A milestone was reached in February 2026, when Math Inc.'s model Gauss "finished" the project in the sense that it produced a sorry-free proof of the optimality of the E_8-lattice in dimension 8, consisting of about 60K lines of Lean code. A week later, they announced similar results for dimension 24, comprising about 180K lines of code. However, their work prompted some public discussion, as the maintainers of the project were unaware that Math Inc. had been working on it until the announcement was made, especially given that they had stopped communicating publicly since November 2025. More backstory on what happened can be found in Jeremy Avigad's essay [3].

Although we now have a sorry-free proof of Viazovska's remarkable result, the project is not yet considered "finished," for several reasons. First, the goal of the project was not to validate Viazovska's proof, since it was never in doubt. Rather, as with other Lean projects, we aim to produce maintainable Lean code that can be upstreamed to Mathlib and prove useful for future formalization projects. For example, the dimension formula for modular forms of level 1, the definition of E_2, the derivative and Serre derivative of modular forms, and Jacobi's identity (2.25) are planned to be upstreamed to Mathlib or have already been upstreamed. The project has also produced several metaprogramming outcomes, including the new tactics norm_numI and tendsto_cont, which are likewise planned to be upstreamed (see Section B.7). The quality of Gauss's code, as well as that of most LLM-generated Lean code, is not yet good enough for this purpose and requires further refactoring. Another aim of the project is to deepen our own understanding of the proof; formalization is an excellent way to do this, as it forces us to scrutinize every detail.

Moreover, LLM-based autoformalization of a natural-language proof does not guarantee that the formalized proof *corresponds to* the informal one, even when the formalized proof is correct. The notion of "correspondence of proofs" is introduced in Simon DeDeo and Eamon Duede's paper [22], and it is highly nontrivial to determine whether two mathematical proofs are essentially the same. In principle, an autoformalized proof may be correct without corresponding to the original proof, and it could be either better or worse than the original. Note that the project does not follow Viazovska's original argument in its entirety [80]. As mentioned above, the proof of the quasimodular form inequalities is based on the approach in Section 5.3 instead of the original argument based on interval arithmetic, and some of the arguments are simplified to make them more amenable to formalization.

Nonetheless, Gauss's code includes formalizations of intermediate results for which we had been searching for the right approach, and it can serve as a useful suggestion for how to handle them. We are currently working to understand and refactor `Gauss`'s code. This also reflects the current state of autoformalization: more developments of this kind will appear in the future, and we will need to adapt to these new technologies.

B.2 Formalizing E_2

The formalization of E_2 was carried out mostly by Chris Birkbeck. Here we briefly explain our choice of definition for E_2 and the proof of its quasimodularity.

The function $E_2 : \mathbb{H} \to \mathbb{C}$ should be holomorphic on $\mathbb{H}$, satisfy the functional equation (2.40), and admit a Fourier expansion of the form (2.39). Here are four possible definitions of E_2:

$$E_2(z) = \frac{1}{2\zeta(2)} \lim_{N\to\infty} \sum_{-N\le m\le N} \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^2} \tag{B.1}$$

$$E_2(z) = 1 - 24 \sum_{n\ge 1} \sigma_1(n) q^n \tag{B.2}$$

$$E_2(z) = (\log \Delta)'(z) = \frac{\Delta'(z)}{\Delta(z)} \tag{B.3}$$

$$E_2(z) = \frac{3E_4'(z) + E_6(z)}{E_4(z)}. \tag{B.4}$$

Note that the last definition implicitly takes Ramanujan's identity (2.43) as a *definition*. Each formulation has its own advantages and disadvantages when proving equivalences between them. For example, if we take (B.3) as the definition, then one must first decide how to define $\Delta(z)$. If we *define* Δ as $\frac{E_4^3 - E_6^2}{1728}$, then the transformation law (2.40) becomes easy to prove, but showing that $\Delta(z)$ is nonzero on $\mathbb{H}$ becomes difficult. Conversely, if we *define* Δ via the q-expansion $q\prod_{n\ge1}(1-q^n)^{24}$, then the nonvanishing of $\Delta(z)$ on $\mathbb{H}$ and the q-expansion (B.2) follow almost immediately, but the transformation law (2.40) becomes difficult. Similarly, one of Ramanujan's identities follows immediately from (B.4) if taken as a *definition*, but this is inconvenient because $E_4(z)$ vanishes at the points equivalent to $e^{2\pi i/3}$ under the $\mathrm{SL}_2(\mathbb{Z})$-action.

In the end, we chose the first definition, and the other three identities above are proved as theorems. One must first handle the conditional convergence of the double sum in the first definition. Once this is done, the q-expansion of E_2 (B.2) follows from the standard argument using the Mittag–Leffler expansion of the cotangent function, $\pi\cot(\pi z) = \frac{1}{z} + \sum_{n>0}\left(\frac{1}{z-n} + \frac{1}{z+n}\right)$, while the functional equation (2.40) can be proved by a direct computation from the definition. The discriminant form $\Delta(z)$ is defined as $\Delta(z) = \eta(z)^{24}$, where $\eta(z) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ is the Dedekind eta function. One can use (B.2) to prove that $E_2(z) = \frac{1}{24}(\log\eta)'(z)$, which implies the transformation law of $\eta(z)$, namely

$$\eta\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{i}}\eta(z),$$

and also implies the transformation law of $\Delta(z)$, ultimately showing that $\Delta(z)$ is a cusp form of weight 12 and level 1.

We also decided not to formalize the full definition of quasimodular forms, since it is not needed for the sphere packing project. However, we plan to formalize the theory of quasimodular forms in the future, and it would be interesting to formalize their applications in several areas of mathematics, such as mirror symmetry [24, 42], partitions [6, 83], and Gromov–Witten invariants [60].

B.3 Schwartzness of Fourier eigenfunctions

The Cohn–Elkies bound [13] holds for Schwartz functions, or more generally for admissible functions; a function $f : \mathbb{R}^d \to \mathbb{C}$ is called *admissible* if there exists a constant $\delta > 0$ such that $|f(\mathbf{x})|$ and $|\widehat{f}(\mathbf{x})|$ are bounded by $C(1+|\mathbf{x}|)^{-d-\delta}$ for some constant $C > 0$. In [80], Viazovska used the bounds (5.10) and (5.11) on the coefficients of the relevant modular forms to establish the Schwartzness of the Fourier eigenfunctions, by proving

$$|\phi_0(z)| \leq C_0 e^{-2\pi\Im z} \tag{B.5}$$

$$|\psi_S(z)| \leq C_S e^{-\pi\Im z} \tag{B.6}$$

for some constants $C_0, C_S > 0$ and all $\Im z > 1/2$. As mentioned above, the proofs of (5.10) and (5.11) rely on the Hardy–Ramanujan circle method or on nonholomorphic Poincaré series, both of which are highly nontrivial to formalize. Instead, we provide a much simpler argument that requires only the product formula for the discriminant form (2.10) and the polynomial growth of the Fourier coefficients of the modular forms appearing in the “numerators.”

Lemma B.3.1. Let $F(z)$ be a holomorphic function on $\mathbb{H}$ with Fourier expansion

$$F(z) = \sum_{n \geq n_0} a_n e^{\pi i n z} \tag{B.7}$$

where $a_{n_0} \neq 0$. Assume that $\sum_{n \geq n_0} |a_n| e^{-\pi n \Im z}$ converges absolutely for $\Im z \geq 1/2$ (for example, when a_n has polynomial growth). Then there exists a constant $C_F > 0$ such that

$$\left| \frac{F(z)}{\Delta(z)} \right| \leq C_F e^{-\pi(n_0-2)\Im z} \tag{B.8}$$

for all z with $\Im z \geq 1/2$.

Proof. By the product formula (2.10),

$$\begin{aligned}
\left| \frac{F(z)}{\Delta(z)} \right| &= \left| \frac{\sum_{n \geq n_0} a_n e^{\pi i n z}}{e^{2\pi i z} \prod_{n \geq 1} (1 - e^{2\pi i n z})^{24}} \right| \\
&= |e^{\pi i (n_0 - 2) z}| \cdot \frac{|\sum_{n \geq n_0} a_n e^{\pi i (n - n_0) z}|}{\prod_{n \geq 1} |1 - e^{2\pi i n z}|^{24}} \\
&\leq e^{-\pi(n_0-2)\Im z} \cdot \frac{\sum_{n \geq n_0} |a_n| e^{-\pi(n-n_0)\Im z}}{\prod_{n \geq 1} (1 - e^{-2\pi n \Im z})^{24}} \\
&\leq e^{-\pi(n_0-2)\Im z} \cdot \frac{\sum_{n \geq n_0} |a_n| e^{-\pi(n-n_0)/2}}{\prod_{n \geq 1} (1 - e^{-\pi n})^{24}} \\
&= C_f \cdot e^{-\pi(n_0-2)\Im z}
\end{aligned}$$

where

$$C_f = \frac{\sum_{n \geq n_0} |a_n| e^{-\pi(n-n_0)/2}}{\prod_{n \geq 1} (1 - e^{-\pi n})^{24}}. \tag{B.9}$$

The denominator also converges; it is simply $e^{\pi} \cdot \Delta(i/2)$. ◻

Corollary B.3.2. The inequalities (B.5) and (B.6) hold.

Proof. Apply Lemma B.3.1, where

$$\begin{aligned}
\phi_0(z) &= \frac{(E_2(z)E_4(z) - E_6(z))^2}{\Delta(z)} = \frac{720^2 e^{4\pi i z} + O(e^{5\pi i z})}{\Delta(z)} \\
\psi_S(z) &= -\frac{H_2^3(2H_2^2 + 5H_2H_4 + 5H_4^2)}{2\Delta(z)} = -\frac{10240 e^{3\pi i z} + O(e^{4\pi i z})}{\Delta(z)}.
\end{aligned}$$

Note that all the coefficients of the numerators are nonnegative. □

Lemma B.3.1 was formalized by Sidharth Hariharan earlier during his master's thesis; see [81].

```
variable (z : ℍ) (hz : 1 / 2 < z.im)
variable (c : ℤ → ℂ) (n₀ : ℤ) (hcn₀ : c n₀ ≠ 0)
variable (hcsum : Summable fun (i : ℕ) ↦ (fouterm c z (i + n₀)))
variable (k : ℕ) (hpoly : c =O[atTop] (fun n ↦ (n ^ k : ℝ)))
variable (f : ℍ → ℂ) (hf : ∀ x : ℍ, f x = Σ' (n : ℕ), (fouterm c x (n + n₀)))

noncomputable def DivDiscBound : ℝ :=
  (Σ' (n : ℕ), norm (c (n + n₀)) * rexp (-π * n / 2)) /
  (∏' (n : ℕ+), (1 - rexp (-π * n)) ^ 24)

include f hf z hz c n₀ hcsum k hpoly in
theorem DivDiscBoundOfPolyFourierCoeff : norm ((f z) / (Δ z)) ≤
  (DivDiscBound c n₀) * rexp (-π * (n₀ - 2) * z.im) := ...

theorem norm_φ₀_le : ∃ C₀ > 0, ∀ z : ℍ, 1 / 2 < z.im →
    norm (φ₀ z) ≤ C₀ * rexp (-2 * π * z.im) := ...
```

B.4 Quasimodular form inequalities

As mentioned earlier, we chose to formalize the proof of the quasimodular form inequalities given in Chapter 5. To do so, we formalized the definitions of derivatives and Serre derivatives, and developed APIs for restricting quasimodular forms to the imaginary axis.

Derivative, Serre derivative, and Ramanujan's formula

We formalized the derivative $D = \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}z}$ and the Serre derivative ∂_k as follows. Note that the derivative is defined to be 0 if a given function is not differentiable.

```
noncomputable def D (F : ℍ → ℂ) : ℍ → ℂ :=
  fun (z : ℍ) => (2 * π * I)⁻¹ * ((deriv (F ∘ ofComplex)) z)
```

```
noncomputable def serre_D (k : ℂ) : (ℍ → ℂ) → (ℍ → ℂ) :=
  fun (F : ℍ → ℂ) => (fun z => D F z - k * 12⁻¹ * E₂ z * F z)
```

Then we proved basic properties of `D` and `serre_D`, including linearity, the Leibniz rule, and the fact that `serre_D` preserves modularity. Eventually, we showed that $\partial_k F$ is a modular form of weight $k+2$ if F is a modular form of weight k.

```
theorem D_mul (F G : ℍ → ℂ) (hF : MDiff F) (hG : MDiff G) : D (F * G) = D F * G + F * D G
    := ...

theorem serre_D_mul (k₁ k₂ : ℤ) (F G : ℍ → ℂ) (hF : MDiff F) (hG : MDiff G) :
    serre_D (k₁ + k₂) (F * G) = (serre_D k₁ F) * G + F * (serre_D k₂ G) := ...

theorem serre_D_slash_equivariant (k : ℤ) (F : ℍ → ℂ) (hF : MDiff F) :
    ∀ γ : SL(2, ℤ), serre_D k F ∣[k + 2] γ = serre_D k (F ∣[k] γ) := ...

theorem serre_D_slash_invariant (k : ℤ) (F : ℍ → ℂ) (hF : MDiff F)
    (γ : SL(2, ℤ)) (h : F ∣[k] γ = F) :
    serre_D k F ∣[k + 2] γ = serre_D k F := ...

noncomputable def serre_D_ModularForm (k : ℤ) (f : ModularForm (Gamma 1) k) :
    ModularForm (Gamma 1) (k + 2) where ...
```

Combined with the dimension formula for modular forms of level 1, Ramanujan's identities (2.42)–(2.44) are also formalized. To prove them, we use `serre_D_slash_invariant` to show that $\partial_1 E_2$, $\partial_4 E_4$, and $\partial_6 E_6$ are modular forms of weight 4, 6, and 8 respectively, and then apply the dimension formula to conclude that they must be scalar multiples of E_4, E_6, and E_4^2, where the constants can be determined by comparing limits as $\Im z \to \infty$. This was largely done with the help of Cameron Freer, who used `Claude` with his `lean4-skills`[1].

```
theorem ramanujan_E₂' : serre_D 1 E₂ = - 12⁻¹ * E₄.toFun := ...
```

[1] https://github.com/cameronfreer/lean4-skills

```
theorem ramanujan_E₄' : serre_D 4 E₄.toFun = - 3⁻¹ * E₆.toFun := ...

theorem ramanujan_E₆' : serre_D 6 E₆.toFun = - 2⁻¹ * E₄.toFun * E₄.toFun := ...

theorem ramanujan_E₂ : D E₂ = 12⁻¹ * (E₂ * E₂ - E₄.toFun) := ...

theorem ramanujan_E₄ : D E₄.toFun = 3⁻¹ * (E₂ * E₄.toFun - E₆.toFun) := ...

theorem ramanujan_E₆ : D E₆.toFun = 2⁻¹ * (E₂ * E₆.toFun - E₄.toFun * E₄.toFun) :=
```

Restriction to the imaginary axis

We defined an API for restricting quasimodular forms to the imaginary axis, which is useful for formalizing the quasimodular form inequalities.

```
noncomputable def ResToImagAxis (F : ℍ → ℂ) : ℝ → ℂ :=
  fun t => if ht : 0 < t then F ⟨(I * t), by simp [ht]⟩ else 0

noncomputable def ResToImagAxis.Real (F : ℍ → ℂ) : Prop :=
  ∀ t : ℝ, 0 < t → (F.resToImagAxis t).im = 0

noncomputable def ResToImagAxis.Pos (F : ℍ → ℂ) : Prop :=
  ResToImagAxis.Real F ∧ ∀ t : ℝ, 0 < t → 0 < (F.resToImagAxis t).re
```

Using these, we formalized the results in Chapter 3, including Proposition 3.2.2 and Corollary 3.3.2 as below.

```
theorem antiDerPos {F : ℍ → ℂ} (hFderiv : MDiff F)
    (hFepos : ResToImagAxis.EventuallyPos F) (hDF : ResToImagAxis.Pos (D F)) :
    ResToImagAxis.Pos F := ...

theorem antiSerreDerPos {F : ℍ → ℂ} {k : ℤ} (hMD : MDiff F)
    (hSDF : ResToImagAxis.Pos (serre_D k F))
    (hF : ResToImagAxis.EventuallyPos F) : ResToImagAxis.Pos F := ...
```

Inequalities

With the above formalization, we can formalize the proof in Section 5.3. The current formalization was done before Gauss's contribution (again with the help of Cameron Freer); Gauss's formalization subsequently followed the same proof.

```
noncomputable def F := (E₂ * E₄.toFun - E₆.toFun) ^ 2
noncomputable def G := H₂ ^ 3 * ((2 : ℝ) · H₂ ^ 2 + (5 : ℝ) · H₂ * H₄ + (5 : ℝ) · H₄ ^ 2)
noncomputable def negDE₂ := - (D E₂)
noncomputable def L₁₀ := (D F) * G - F * (D G)
noncomputable def SerreDer_22_L₁₀ := serre_D 22 L₁₀
noncomputable def FReal (t : ℝ) : ℝ := (F.resToImagAxis t).re
noncomputable def GReal (t : ℝ) : ℝ := (G.resToImagAxis t).re
noncomputable def FmodGReal (t : ℝ) : ℝ := (FReal t) / (GReal t)

theorem MLDE_F : serre_D 12 (serre_D 10 F) =
    5 * 6⁻¹ * E₄.toFun * F + 7200 * Δ_fun * negDE₂ := ...

theorem MLDE_G : serre_D 12 (serre_D 10 G) =
    5 * 6⁻¹ * E₄.toFun * G - 640 * Δ_fun * H₂ := ...

private theorem serre_D_L₁₀_pos_imag_axis : ResToImagAxis.Pos SerreDer_22_L₁₀ := ...

theorem FmodG_strictAntiOn : StrictAntiOn FmodGReal (Set.Ioi 0) := ...

theorem FG_inequality_1 {t : ℝ} (ht : 0 < t) :
    FReal t + 18 * (π ^ (-2 : ℤ)) * GReal t > 0 := ...

theorem FG_inequality_2 {t : ℝ} (ht : 0 < t) :
    FReal t - 18 * (π ^ (-2 : ℤ)) * GReal t < 0 := ...
```

B.5 Jacobi theta functions and transformation laws

We first needed to formalize the Jacobi theta functions $\Theta_2, \Theta_3, \Theta_4$ and H_2, H_3, H_4. Fortunately, the two-variable Jacobi theta function

$$\theta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{2\pi i n z + \pi i n^2 \tau}$$

already exists in `mathlib` as `jacobiTheta₂`, formalized by David Loeffler, so we can define $\Theta_2, \Theta_3, \Theta_4$ and H_2, H_3, H_4 as specializations of $\theta(z, \tau)$:

$$\begin{aligned}\Theta_2(\tau) &= e^{\pi i \tau/4}\theta(\tau/2, \tau),\\ \Theta_3(\tau) &= \theta(0, \tau),\\ \Theta_4(\tau) &= \theta(1/2, \tau).\end{aligned}$$

This is helpful for computing the S-action on the theta functions (2.31), since the functional equation for $\theta(z, \tau)$,

$$\theta(z, \tau) = \frac{1}{(-i\tau)^{1/2}} e^{-\pi i z^2/\tau} \theta(z/\tau, -1/\tau),$$

is already formalized in `mathlib` as `jacobiTheta₂_functional_equation`. Using (2.30) and (2.31), we can prove that H_2, H_3, H_4 are invariant under the slash actions of $\alpha = T^2$, $\beta = -ST^{-2}S$, and $-I$, which together generate $\Gamma(2)$. After establishing holomorphicity (`MDifferentiable`) and the growth condition (boundedness at the cusps), we conclude that H_2, H_3, H_4 are modular forms of weight 2 and level $\Gamma(2)$.

```
noncomputable def Θ₂_term (n : ℤ) (τ : ℍ) : ℂ := cexp (π * I * (n + 1 / 2 : ℂ) ^ 2 * τ)
noncomputable def Θ₃_term (n : ℤ) (τ : ℍ) : ℂ := cexp (π * I * (n : ℂ) ^ 2 * τ)
noncomputable def Θ₄_term (n : ℤ) (τ : ℍ) : ℂ := (-1) ^ n * cexp (π * I * (n : ℂ) ^ 2 * τ)
noncomputable def Θ₂ (τ : ℍ) : ℂ := Σ' n : ℤ, Θ₂_term n τ
noncomputable def Θ₃ (τ : ℍ) : ℂ := Σ' n : ℤ, Θ₃_term n τ
noncomputable def Θ₄ (τ : ℍ) : ℂ := Σ' n : ℤ, Θ₄_term n τ
noncomputable def H₂ (τ : ℍ) : ℂ := (Θ₂ τ) ^ 4
noncomputable def H₃ (τ : ℍ) : ℂ := (Θ₃ τ) ^ 4
noncomputable def H₄ (τ : ℍ) : ℂ := (Θ₄ τ) ^ 4

noncomputable def H₂_MF : ModularForm (Γ 2) 2 := {...}
noncomputable def H₃_MF : ModularForm (Γ 2) 2 := {...}
noncomputable def H₄_MF : ModularForm (Γ 2) 2 := {...}
```

B.6 Reducing level 2 modular form identities to level 1

To prove the Jacobi identity (2.25) or the identities on the derivatives of the theta functions (2.49), (2.50), (2.51), one standard approach is to use the fact that the Serre derivative preserves depth, and then to inspect the first few q-coefficients to express the relevant forms in terms of a basis of $\mathcal{S}_4(\Gamma(2))$. However, computing $\dim_{\mathbb{C}} \mathcal{S}_4(\Gamma(2))$ formally would require considerable additional work, as the valence formula is not yet available in `mathlib`. Instead, we employ a few tricks to reduce these statements to level 1, whose dimension formulas are already formalized.

Lemma B.6.1 (Jacobi's identity)**.** We have $H_2 + H_4 = H_3$.

Proof. Let $f = (H_2 + H_4 - H_3)^2$. Clearly, f is a modular form of weight 4 and level $\Gamma(2)$. Using the transformation rules for H_2, H_3, H_4, one computes

$$f|_S = (-H_4 - H_2 + H_3)^2 = f, \tag{B.10}$$

$$f|_T = (-H_2 + H_3 - H_4)^2 = f, \tag{B.11}$$

so f is in fact a modular form of level 1. Taking the limit as $z \to i\infty$ shows that f is a cusp form, and therefore $f = 0$, since there is no nonzero cusp form of weight 4 and level 1. □

The identities (2.32), (2.33), and (2.34) can be proved by similar tricks. The right-hand sides of (2.32), (2.33), and (2.34) are modular forms of level 1 and of the desired weights, and the form in (2.34) is a cusp form since H_2 is. The identities then follow from the dimension calculations $\dim \mathcal{M}_4(1) = \dim \mathcal{M}_6(1) = \dim \mathcal{S}_{12}(1) = 1$ and a comparison of the first nonzero q-coefficients.

```
theorem jacobi_identity : H₂ + H₄ = H₃ := ...
```

Proposition B.6.2. The equations (2.46)–(2.48) and (2.49)–(2.51) hold.

Proof. The equivalences are immediate from the definition of the Serre derivative. Define f_2, f_3, f_4 as the differences of the left- and right-hand sides of (2.46), (2.47), and (2.48), respectively:

$$f_2 := \partial_2 H_2 - \frac{1}{6} H_2(H_2 + 2H_4),$$

$$f_3 := \partial_2 H_3 - \frac{1}{6}(H_2^2 - H_4^2),$$
$$f_4 := \partial_2 H_4 + \frac{1}{6}H_4(2H_2 + H_4).$$

A priori, these are modular forms of weight 4 and level $\Gamma(2)$; our goal is to show that they vanish. By Jacobi's identity (2.25), $f_2 + f_4 = f_3$. Moreover, the transformation rules for H_2, H_3, H_4 yield

$$f_2|_S = -f_4,$$
$$f_2|_T = -f_2,$$
$$f_4|_S = -f_2,$$
$$f_4|_T = f_3 = f_2 + f_4.$$

Now define

$$g := (2H_2 + H_4)f_2 + (H_2 + 2H_4)f_4,$$
$$h := f_2^2 + f_2 f_4 + f_4^2.$$

One can check that both g and h are invariant under the actions of S and T, so they are modular forms of level 1. Analyzing the limits of g and h as $z \to i\infty$ shows that they are cusp forms, and hence $g = h = 0$, since $\dim_{\mathbb{C}} \mathcal{S}_6(1) = \dim_{\mathbb{C}} \mathcal{S}_8(1) = 0$. This yields

$$\begin{aligned} 3E_4 f_2^2 &= 3(H_2^2 + H_2H_4 + H_4^2) f_2^2 \\ &= \left((2H_2 + H_4)^2 - (2H_2 + H_4)(H_2 + 2H_4) + (H_2 + 2H_4)^2\right) f_2^2 \\ &= (2H_2 + H_4)^2 (f_2^2 + f_2 f_4 + f_4^2) = 0, \end{aligned}$$

which implies $f_2 = 0$. □

```
theorem D_H₂ :
    D H₂ = (1 / 6 : ℂ) • (H₂ ^ 2 + (2 : ℂ) • (H₂ * H₄)) + (1 / 6 : ℂ) • (E₂ * H₂) := ...

theorem D_H₃ :
    D H₃ = (1 / 6 : ℂ) • (H₂ ^ 2 - H₄ ^ 2) + (1 / 6 : ℂ) • (E₂ * H₃) := ...

theorem D_H₄ :
    D H₄ = (-(1 / 6 : ℂ)) • ((2 : ℂ) • (H₂ * H₄) + H₄ ^ 2) + (1 / 6 : ℂ) • (E₂ * H₄) := ...
```

B.7 Metaprogramming outcomes

Working on the sphere packing project produced several interesting metaprogramming outcomes, which we expect to be useful for future projects.

`norm_numI`

`norm_numI` is a normalization-simplification procedure to bring `norm_num`-like functionality to the complex numbers. The tactic recursively parses an expression in $\mathbb{C}$, splits all atoms into their real and imaginary parts, calls `norm_num` on their real and imaginary parts, and recombines them using the rules of complex arithmetic to return a normalized, simplified expression in $\mathbb{C}$. The key insight was identifying that the right 'normal form' for an expression in $\mathbb{C}$ is $a + bi$, where $a, b \in \mathbb{R}$ are both in normal form (in the sense of `norm_num`). This was mostly done by Edison Xie, with input from Heather Macbeth, Bhavik Mehta, and Sidharth Hariharan.

`tendsto_cont`

The second tactic developed was `tendsto_cont`, which uses the continuity of the projection maps from a product of topological spaces to prove goals of the form `Tendsto (fun z => expr(f₁ z, ..., fₙ z)) l (nhds c)` where atomic limits `Tendsto fᵢ l (nhds aᵢ)` are known from context and `expr` is continuous in the atoms (proved via `fun_prop`). We use it to simplify the proof of the limit of quasimodular forms that are polynomials in E_2, E_4, E_6, H_2, H_4. This was mostly done by Cameron Freer using `Claude`, with input from Bhavik Mehta and myself.

```
lemma H₂_resToImagAxis_tendsto_zero : Tendsto H₂.resToImagAxis atTop (nhds 0) := ...

lemma H₄_resToImagAxis_tendsto_one : Tendsto H₄.resToImagAxis atTop (nhds 1) := ...

-- Before ‘tendsto_cont‘
lemma denominator_tendsto_at_infty :
    Tendsto (fun s ↦ (H₄.resToImagAxis s).re ^ 3 *
      (2 * (H₄.resToImagAxis s).re ^ 2 + 5 * (H₂.resToImagAxis s).re * (H₄.resToImagAxis s).re
        + 5 * (H₂.resToImagAxis s).re ^ 2)) atTop (nhds 2) := by
  have hH₂_lim := H₂_re_resToImagAxis_tendsto_zero
```

```
  have hH₄_lim := H₄_re_resToImagAxis_tendsto_one
  have hlim : (1 : ℝ) ^ 3 * (2 * 1 ^ 2 + 5 * 0 * 1 + 5 * 0 ^ 2) = 2 := by norm_num
  convert (hH₄_lim.pow 3).mul ((hH₄_lim.pow 2 |>.const_mul 2).add
    ((hH₂_lim.mul hH₄_lim |>.const_mul 5).add (hH₂_lim.pow 2 |>.const_mul 5))) using 1
  · ext s; ring
  · ring

-- After ‘tendsto_cont‘
lemma denominator_tendsto_at_infty :
    Tendsto (fun s ↦ (H₄.resToImagAxis s).re ^ 3 *
      (2 * (H₄.resToImagAxis s).re ^ 2 + 5 * (H₂.resToImagAxis s).re * (H₄.resToImagAxis s).re
        + 5 * (H₂.resToImagAxis s).re ^ 2)) atTop (nhds 2) := by tendsto_cont
    [H₂_resToImagAxis_tendsto_zero, H₄_resToImagAxis_tendsto_one]
```

---

fun_prop

fun_prop is an existing Lean 4 tactic for establishing properties of functions, such as continuity and differentiability. We found it very effective when dealing with properties of restrictions of quasimodular forms to the imaginary axis, such as realness and positivity.

```
-- Before ‘fun_prop‘
theorem F_imag_axis_real : ResToImagAxis.Real F := by
  unfold F
  have hProd : ResToImagAxis.Real (E₂ * E₄.toFun) :=
    ResToImagAxis.Real.mul E₂_imag_axis_real E₄_imag_axis_real
  have hNeg : ResToImagAxis.Real ((-1 : ℝ) · E₆.toFun) :=
    ResToImagAxis.Real.smul E₆_imag_axis_real
  have hSub : ResToImagAxis.Real (E₂ * E₄.toFun - E₆.toFun) := by
    have hEq : E₂ * E₄.toFun - E₆.toFun = E₂ * E₄.toFun + (-1 : ℝ) · E₆.toFun := by
      ext z
      simp [sub_eq_add_neg]
    simpa [hEq] using ResToImagAxis.Real.add hProd hNeg
  simpa [pow_two] using ResToImagAxis.Real.mul hSub hSub

-- After ‘fun_prop‘
```

```
theorem F_imag_axis_real : ResToImagAxis.Real F := by unfold F; fun_prop
```

We also found it useful for proving the holomorphicity (`MDifferentiable`) of quasimodular forms. We registered `MDifferentiable` and theorems like `MDifferentiable.add` or `MDifferentiable.mul`, along with holomorphicity of E_2, E_4, E_6, H_2, H_4, as `fun_prop` lemmas, so that we can prove holomorphicity of more complicated quasimodular forms by simply writing `fun_prop` after unfolding the definitions. It was first suggested by Jujian Zhang to use `fun_prop` to `MDifferentiable`, where we found that it is also useful for `ResToImagAxis` as mentioned above.

```
-- Before 'fun_prop'
theorem F_aux : ... := by  -- some auxiliary identity
  ...
  -- Holomorphicity of the terms
  · exact E₂_holo'
  · exact E₄.holo'
  · exact MDifferentiable.mul E₂_holo' E₄.holo'
  · exact E₆.holo'
  · exact MDifferentiable.sub (MDifferentiable.mul E₂_holo' E₄.holo') E₆.holo'

-- After 'fun_prop'
theorem F_aux : ... := by  -- some auxiliary identity
  ...
  -- Holomorphicity of the terms
  repeat fun_prop
```

Appendix C

Tables

s	w	$X_{w,s}$
1	6	$\frac{E_2E_4 - E_6}{720} = q + 18q^2 + 84q^3 + 292q^4 + 630q^5 + \cdots$
	8	$\frac{-E_2E_6 + E_4^2}{1008} = q + 66q^2 + 732q^3 + 4228q^4 + 15630q^5 + \cdots$
	10	$\frac{E_2E_4^2 - E_4E_6}{720} = q + 258q^2 + 6564q^3 + 66052q^4 + 390630q^5 + \cdots$
	12	$\frac{-12E_2E_4E_6 + 5E_4^3 + 7E_6^2}{3991680} = q^2 + 56q^3 + 1002q^4 + 9296q^5 + 57708q^6 + \cdots$
	14	$\frac{7E_2E_4^3 + 5E_2E_6^2 - 12E_4^2E_6}{4717440} = q^2 + 128q^3 + 4050q^4 + 58880q^5 + 525300q^6 + \cdots$
2	4	$\frac{-E_2^2 + E_4}{288} = q + 6q^2 + 12q^3 + 28q^4 + 30q^5 + \cdots$
	8	$\frac{-7E_2^2E_4 + 2E_2E_6 + 5E_4^2}{362880} = q^2 + 16q^3 + 102q^4 + 416q^5 + 1308q^6 + \cdots$
	10	$\frac{5E_2^2E_6 + 2E_2E_4^2 - 7E_4E_6}{1088640} = q^2 + \frac{104}{3}q^3 + 390q^4 + 2480q^5 + 11140q^6 + \cdots$
	12	$\frac{-77E_2^2E_4^2 + 34E_2E_4E_6 + 50E_4^3 - 7E_6^2}{798336000} = q^3 + \frac{51}{2}q^4 + \frac{1422}{5}q^5 + 1944q^6 + 9714q^7 + \cdots$
	14	$\frac{13E_2^2E_4E_6 + E_2E_4^3 - 3E_2E_6^2 - 11E_4^2E_6}{415134720} = q^3 + \frac{93}{2}q^4 + 810q^5 + 8004q^6 + 54474q^7 + \cdots$

Table C.1: Extremal quasimodular forms of depth ≤ 2 and weight ≤ 14.

w	N	normalized extremal form
2	2	$\frac{-E_2 + A_2}{48} = q + 2q^2 + 4q^3 + 4q^4 + 6q^5 + \cdots$
4		$\frac{-E_2A_2 + A_{4,0}}{864} = q^2 + \frac{8}{3}q^3 + 10q^4 + 16q^5 + \frac{100}{3}q^6 + \cdots$
6		$\frac{-5E_2A_2^2 - 3E_2A_{4,0} - 3A_2^3 + 11A_2A_{4,0}}{115200} = q^3 + 3q^4 + \frac{78}{5}q^5 + 32q^6 + 90q^7 + \cdots$
8		$\frac{-20E_2A_2^3 - 52E_2A_2A_{4,0} - 27A_2^4 + 74A_2^2A_{4,0} + 25A_{4,0}^2}{16934400} = q^4 + \frac{16}{5}q^5 + \frac{104}{5}q^6 + \frac{1728}{35}q^7 + \cdots$
2	3	$\frac{-E_2 + B_2}{36} = q + 3q^2 + 3q^3 + 7q^4 + 6q^5 + 9q^6 + \cdots$
4		$\frac{-2E_2B_2 - B_2^2 + 3B_{4,0}}{432} = q^2 + 8q^3 + 15q^4 + 32q^5 + 75q^6 + 96q^7 + \cdots$
6		$\frac{-5E_2B_2^2 - 7E_2B_{4,0} - 5B_2^3 + B_2B_{4,0} + 16B_{6,0}}{129600} = q^4 + \frac{18}{5}q^5 + 6q^6 + 22q^7 + 45q^8 + 66q^9 + \cdots$
8		$\frac{-35E_2B_2^3 - 84E_2B_2B_{4,0} - 25E_2B_{6,0} - 273B_2^2B_{4,0} + 447B_2B_{6,0} - 30B_{4,0}^2}{11430720} = q^5 + 7q^6 + \frac{117}{7}q^7 + \cdots$
2	4	$\frac{-E_2 + C_{2,0} - 24C_{2,1}}{96} = q^2 + 2q^4 + 4q^6 + 4q^8 + 6q^{10} + \cdots$
4		$\frac{-E_2C_{2,0} + C_{2,0}^2 - 24C_{2,0}C_{2,1} - 96C_{2,1}^2}{1728} = q^4 + \frac{8}{3}q^6 + 10q^8 + 16q^{10} + \frac{100}{3}q^{12} + \cdots$
6		$\frac{-E_2C_{2,0}^2 + 18E_2C_{2,1}^2 + C_{2,0}^3 - 24C_{2,0}^2C_{2,1} - 114C_{2,0}C_{2,1}^2 + 432C_{2,1}^3}{28800} = q^6 + 3q^8 + \frac{78}{5}q^{10} + 32q^{12} + 90q^{14} + \cdots$
8		$\frac{-3E_2C_{2,0}^3 + 104E_2C_{2,0}C_{2,1}^2 + 3C_{2,0}^4 - 72C_{2,0}^3C_{2,1} - 392C_{2,0}^2C_{2,1}^2 + 2496C_{2,0}C_{2,1}^3 + 4800C_{2,1}^4}{1411200} = q^8 + \frac{16}{5}q^{10} + \cdots$

Table C.2: Extremal quasimodular forms of depth 1, weight ≤ 8, and level $\Gamma_0(N)$ for $N = 2, 3, 4$.

Appendix D

Completely positive quasimodular forms of higher levels

In Section 6.4, we constructed positive quasimodular forms of level > 1 by using the monotonicity of the functions of the form in (6.1). Here we give some examples of *completely* positive quasimodular forms of level > 1, directly constructed from level 1 quasimodular forms by analyzing the Fourier coefficients.

Let $F(z) = \sum_{n\geq 1} a_n q^n$ be a quasimodular form of level 1. Then $F(z) - cF(Nz) = \sum_{n\geq 1}(a_n - c a_{n/N})q^n$ is completely positive if and only if

$$\inf_{n\geq 1} \frac{a_{Nn}}{a_n} \geq c. \tag{D.1}$$

In particular, the infimum (D.1) is the largest possible constant c such that $F(z) - cF(Nz)$ remains completely positive.

Proposition D.0.1. For given weight w and depth s, define

$$Y_{w,s}(z) := X_{w,s}(z) - 2^{w-s} X_{w,s}(2z).$$

Then $Y_{4,2}, Y_{8,2}, Y_{10,2}$, and $Y_{12,2}$ are completely positive.

Proof. Let $a_{w,n}$ be the n-th Fourier coefficient of $X_{w,2}$. It is enough to show that

$$\inf_{n} \frac{a_{w,2n}}{a_{w,n}} \geq 2^{w-2}. \tag{D.2}$$

We can express the extremal forms in terms of (derivatives of) Eisenstein series and obtain explicit formulas for the Fourier coefficients as

$$X_{4,2} = -\frac{E_2'}{24} = \sum_{n\geq 1} n\sigma_1(n)q^n, \tag{D.3}$$

$$X_{8,2} = -\frac{E_6'}{15120} - \frac{E_4''}{7200} = \sum_{n\geq 2}\left(\frac{n\sigma_5(n) - n^2\sigma_3(n)}{30}\right)q^n, \tag{D.4}$$

$$X_{10,2} = \frac{(E_4^2)'}{60480} + \frac{E_6''}{63504} = \sum_{n\geq 2}\left(\frac{n\sigma_7(n) - n^2\sigma_5(n)}{126}\right)q^n. \tag{D.5}$$

(Note that $E_4^2 = 1 + 480\sum_{n\geq 1}\sigma_7(n)q^n$ is the weight 8 normalized Eisenstein series.) Using these, we can compute the infimum (D.2) precisely. Recall that the divisor sum functions are multiplicative. Write $n = 2^k m$ for $k \geq 0$ and odd m. Then the infimum for $w = 4$ can be computed as

$$\inf_{n\geq 1}\frac{2n\sigma_1(2n)}{n\sigma_1(n)} = \inf_{n\geq 1}\frac{2^{k+1}m\sigma_1(2^{k+1})\sigma_1(m)}{2^k m\sigma_1(2^k)\sigma_1(m)} = \inf_{k\geq 0}\frac{2(2^{k+2}-1)}{2^{k+1}-1} = 4 = 2^{4-2}.$$

The cases $w = 8$ and $w = 10$ are similar but slightly more complicated. When $w = 8$, we first observe that $n\sigma_5(n) - n^2\sigma_3(n) \geq 0 \Leftrightarrow b_n := \frac{\sigma_5(n)}{n\sigma_3(n)} \geq 1$ for all $n \geq 2$, because of the complete positivity of $X_{8,2}$[1]. Again, we write $n = 2^k m$ with odd m. Then

$$\begin{aligned}\frac{a_{8,2n}}{a_{8,n}} &= \frac{2n\sigma_5(2n) - 4n^2\sigma_3(2n)}{n\sigma_5(n) - n^2\sigma_3(n)}\\ &= \frac{2^{k+1}m\sigma_5(2^{k+1})\sigma_5(m) - 2^{2k+2}m^2\sigma_3(2^{k+1})\sigma_3(m)}{2^k m\sigma_5(2^k)\sigma_5(m) - 2^{2k}m^2\sigma_3(2^k)\sigma_3(m)}\\ &= \frac{4\sigma_3(2^{k+1})}{\sigma_3(2^k)}\cdot\frac{b_{2^{k+1}}b_m - 1}{b_{2^k}b_m - 1}.\end{aligned}$$

It is easy to check that b_{2^k} is increasing in $k \geq 0$. Hence

$$\frac{a_{8,2n}}{a_{8,n}} \geq \frac{4\sigma_3(2^{k+1})}{\sigma_3(2^k)}\cdot\frac{b_{2^{k+1}}}{b_{2^k}} = \frac{2\sigma_5(2^{k+1})}{\sigma_5(2^k)} = \frac{2(2^{5(k+2)}-1)}{2^{5(k+1)}-1} > 2^6 = 2^{8-2}.$$

The constant 2^6 is also optimal by considering $n = 2^k$ with $k \to \infty$. The same argument also works for $w = 10$.

[1]The inequality also follows from the trivial estimate $n^a \leq \sigma_a(n) \leq 1^a + 2^a + \cdots + n^a$.

For $w = 12$, $X_{12,2}$ can be expressed as

$$X_{12,2} = \frac{1}{18000}\left(\frac{17\Delta}{21} - \frac{E'_{10}}{308} - \frac{E''_8}{288}\right) = \frac{1}{18000}\sum_{n\geq 3}\left(\frac{17\tau(n)}{21} + \frac{6n\sigma_9(n)}{7} - \frac{5n^2\sigma_7(n)}{3}\right)q^n =: \sum_{n\geq 3} c_n q^n,$$

so it is enough to show that $c_{2n} \geq 2^{10}c_n$ for $n \geq 2$, or equivalently,

$$\frac{17\tau(2n)}{21} + \frac{12n\sigma_9(2n)}{7} - \frac{10n^2\sigma_7(2n)}{3} - 2^{10}\left(\frac{17\tau(n)}{21} + \frac{6n\sigma_9(n)}{7} - \frac{5n^2\sigma_7(n)}{3}\right) \geq 0. \tag{D.6}$$

Write $n = 2^k m$ with odd m. Then (D.6) is equivalent to

$$\begin{aligned}
&\frac{17\tau(2^{k+1})\tau(m)}{21} + \frac{12\cdot 2^k m\sigma_9(2^{k+1})\sigma_9(m)}{7} - \frac{5\cdot 2^{2k+2}m^2\sigma_7(2^{k+1})\sigma_7(m)}{3}\\
&\quad - 2^{10}\left(\frac{17\tau(2^k)\tau(m)}{21} + \frac{6\cdot 2^k m\sigma_9(2^k)\sigma_9(m)}{7} - \frac{5\cdot 2^{2k}m^2\sigma_7(2^k)\sigma_7(m)}{3}\right)\\
&= \frac{17\tau(m)}{21}(\tau(2^{k+1}) - 1048\tau(2^k))\\
&\quad + \frac{6\cdot 2^k m\sigma_9(m)}{7}(2\sigma_9(2^{k+1}) - 2^{10}\sigma_9(2^k)) - \frac{5\cdot 2^{2k}m^2\sigma_7(m)}{3}(2^2\sigma_7(2^{k+1}) - 2^{10}\sigma_7(2^k))\\
&= \frac{17\tau(m)}{21}(\tau(2^{k+1}) - 1048\tau(2^k))\\
&\quad + \frac{6\cdot 2^{k+1} m\sigma_9(m)}{7}\left(\frac{2^{9(k+2)}-1}{511} - 512\cdot\frac{2^{9(k+1)}-1}{511}\right)\\
&\quad - \frac{5\cdot 2^{2k+2}m^2\sigma_7(m)}{3}\left(\frac{2^{7(k+2)}-1}{127} - 256\cdot\frac{2^{7(k+1)}-1}{127}\right)\\
&= \frac{5}{3}\cdot\frac{128}{127}m^2\sigma_7(m)2^{2k+2}\left(2^{7(k+1)} - \frac{255}{128}\right)\\
&\quad + \frac{17}{21}\tau(m)\left(\tau(2^{k+1}) - 1048\tau(2^k)\right) + \frac{6}{7}m\sigma_9(m)2^{k+1}
\end{aligned} \tag{D.7}$$

When $k = 0$, $m \geq 3$ (since $n \geq 2$) and (D.7) can be bounded from below by

$$\frac{2520}{3}m^2\sigma_7(m) - \frac{17\cdot 1072}{21}\tau(m) + \frac{12}{7}m\sigma_9(m) > \frac{2520}{3}m^9 - \frac{18224}{21}\sigma_0(m)m^{\frac{11}{2}} + \frac{12}{7}m^{10} \tag{D.8}$$

and the right-hand side can be easily checked to be positive for $m \geq 3$ using the estimate $\sigma_0(m) \leq m$. When $k \geq 1$, since the last term of (D.7) is positive, it is enough to show that

$$m^2\sigma_7(m) > |\tau(m)| \quad\text{and}\quad \frac{5}{3}(512\cdot 2^{9k} - 8\cdot 2^{2k}) > \frac{17}{21}|\tau(2^{k+1}) - 1048\tau(2^k)|$$

for odd m and $k \geq 0$. The first inequality easily follows from Deligne's bound and the trivial estimates $\sigma_7(m) \geq m^7$ and $m \geq \sigma_0(m)$:

$$m^2\sigma_7(m) \geq m^2 \cdot m^7 = m^9 \geq m^{\frac{13}{2}} = m \cdot m^{\frac{11}{2}} \geq \sigma_0(m)m^{\frac{11}{2}} > |\tau(m)|.$$

The second inequality can also be checked by using Deligne's bound $|\tau(2^k)| \leq \sigma_0(2^k)2^{\frac{11}{2}k} = (k+1)2^{\frac{11}{2}k}$:

$$\begin{aligned}
&\frac{5}{3}(512 \cdot 2^{9k} - 8 \cdot 2^{2k}) - \frac{17}{21}(|\tau(2^{k+1})| + 1048|\tau(2^k)|) \\
&\geq \frac{2560}{3} \cdot 2^{9k} - \frac{40}{3} \cdot 2^{2k} - \frac{17}{21}((k+2)2^{\frac{11}{2}(k+1)} + 1048(k+1)2^{\frac{11}{2}k}) \\
&\geq \frac{2560}{3} \cdot 2^{9k} - \frac{40}{3} \cdot 2^{2k} - \frac{17}{21}((2^{\frac{11}{2}} + 1048)k + (2^{\frac{13}{2}} + 1048))2^{\frac{11}{2}k} \qquad \text{(D.9)}
\end{aligned}$$

which can be easily checked to be positive for $k \geq 1$. □

Remark D.0.2. The positivity assertions in (D.8) and (D.9) are verified in Lean 4 with the help of `AxiomProver`.

Using these, we can reprove that the quasimodular form L in Section 6.5 is (completely) positive by writing it as

$$L = a_1' X_{8,2}^{[2]}AB + a_2' Y_{8,2}AB + a_3' X_{10,2}^{[2]}A + a_4' Y_{10,2}A + a_5' X_{12,2}^{[2]} + a_6' Y_{12,2}$$

where

$$\begin{aligned}
a_1' &= 43027200, \\
a_2' &= 550800, \\
a_3' &= 60963840, \\
a_4' &= 116640, \\
a_5' &= 339075072000, \\
a_6' &= 331776000.
\end{aligned}$$

In fact, we conjecture that $Y_{w,2}$ is completely positive for *all* $w \neq 6$.

Conjecture D.0.3. For all $w \neq 6$, $Y_{w,2}(z) = X_{w,2}(z) - 2^{w-2}X_{w,2}(2z)$ is completely positive.

This conjecture is stronger than the Kaneko–Koike conjecture on the complete positivity of $X_{w,2}$, since we can write $X_{w,2}$ as

$$X_{w,2}(z) = Y_{w,2}(z) + 2^{w-2}Y_{w,2}(2z) + 2^{2(w-2)}Y_{w,2}(4z) + 2^{3(w-2)}Y_{w,2}(8z) + \cdots .$$

Here are the q-expansions of $Y_{w,2}$ for $w \leq 16$:

$$\begin{aligned}
Y_{4,2} &= q + 2q^2 + 12q^3 + 4q^4 + 30q^5 + \cdots ,\\
Y_{8,2} &= q^2 + 16q^3 + 38q^4 + 416q^5 + 284q^6 + \cdots ,\\
Y_{10,2} &= q^2 + \frac{104}{3}q^3 + 134q^4 + 2480q^5 + \frac{6796}{3}q^6 + \cdots ,\\
Y_{12,2} &= q^3 + \frac{51}{2}q^4 + \frac{1422}{5}q^5 + 920q^6 + 9714q^7 + \cdots ,\\
Y_{14,2} &= q^3 + \frac{93}{2}q^4 + 810q^5 + 3908q^6 + 54474q^7 + 93279q^8 + \cdots ,\\
Y_{16,2} &= q^4 + \frac{864}{25}q^5 + \frac{2736}{5}q^6 + \frac{188288}{35}q^7 + \frac{107998}{5}q^8 + \frac{1051008}{5}q^9 + \cdots .
\end{aligned}$$

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