Singular Algebraic Curves

Seewoo Lee

This note closely follows Chapter 4 of Serre's *Algebraic Groups and Class Fields*. We generalize the theory in Chapter 2 to *nonsingular* curves via normalization. We also associate a singular curve $X_{\mathfrak{m}}$ to a modulus $\mathfrak{m}$ of a nonsingular curve X, which will be used in the construction of generalized Jacobians later. Most of the theorems (Riemann–Roch, Serre duality) hold when we replace $\mathcal{L}(D)$ and $\underline{\Omega}(D)$ with appropriate notions $\mathcal{L}'(D)$ and $\underline{\Omega}'(D)$ for singular curves, and the genus g with *arithmetic* genus.

1 Structure of a singular algebraic curve

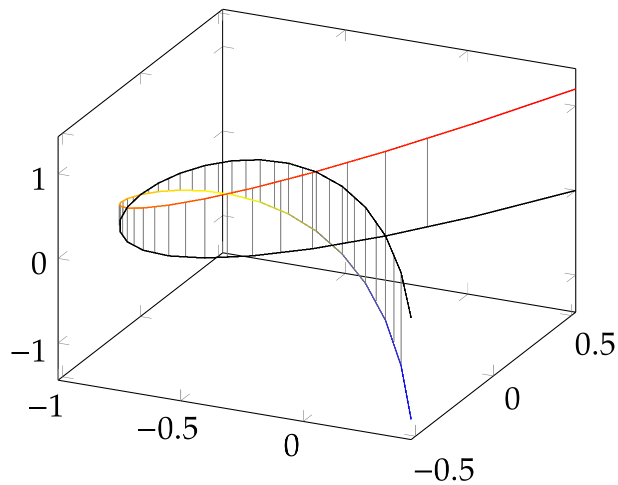
1.1 Normalization of an algebraic variety

For an irreducible algebraic variety X', *normalization* $X \to X'$ can be constructed locally as taking integral closure $\mathcal{O}_Q$ of each $\mathcal{O}'_Q$ in $K = k(X')$. For an affine open $U' \subset X'$ with $A' = \mathcal{O}'(U')$, the corresponding normal affine variety $U \to U'$ corresponds to an integral closure A of A' in K, and *glueing* these U give X. Let $\mathcal{O} = p_*(\mathcal{O}_X)$, whose stalks are $\mathcal{O}_Q = \cap_{P\to Q}\mathcal{O}_P$ and $\mathcal{O}'$ is a subsheaf of $\mathcal{O}$. See this MO answer for the nice geometric intuitions of normal varieties.

Here's an example from Vakil's note: consider a *nodal curve* $X' : y^2 = x^3 + x^2$, singularity at $(0, 0)$. One needs to normalize the ring $R = k[x, y]/(y^2 - x^3 - x^2)$. The element $t = y/x \in \mathrm{Frac}(R) = k(x, y)$ satisfies the integral equation $t^2 = x+1$, and a corresponding variety $X = \{(x, y, t) : y^2 = x^3 + x^2, y = tx, t^2 = x + 1\}$ in $\mathbb{A}^3$ with a projection map $X \to X', (x, y, t) \mapsto (x, y)$. This gives a normalization of X', which is a nonsingular curve (which can be checked by computing it's Jacobian

$$J = \begin{pmatrix} -3x^2 - 2x & 2y & 0 \\ -t & 1 & -x \\ 1 & 0 & -2t \end{pmatrix}$$

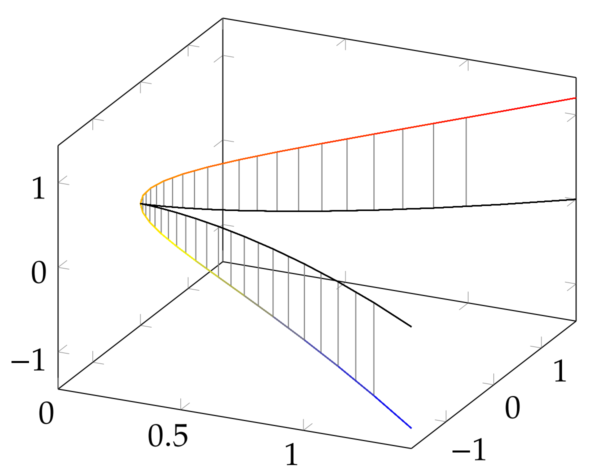
whose determinant is 0 and the first two columns are linearly independent, so has rank $2 = 3-1$ and nonsingular. In fact, $X \simeq \mathbb{A}^1$ since $X = \{(t^2-1, t(t^2-1), t)\}$.



Another example is the curve with a *cusp*: $X' : y^2 = x^3$, singularity at $(0,0)$. We can normalize the ring $R = k[x,y]/(y^2 - x^3)$ similarly as $\widetilde{R} = k[x,y,t]/(y^2 - x^3, t^2 - x, y - tx)$ (taking $t = y/x$), which corresponds to the curve $X = \{(x,y,t) : y^2 = x^3, t^2 = x, y = tx\} = \{(t^2, t^3, t)\}$. It's Jacobian is given by

$$J = \begin{pmatrix} -3x^2 & -2y & 0 \\ 1 & 0 & -2t \\ -t & 1 & -x \end{pmatrix}$$

whose determinant is 0 and the first two columns are linearly independent, so has rank $2 = 3 - 1$. In fact, we have $X = \{(t^2, t^3, t)\} \simeq \mathbb{A}^1$.



We denote as $\mathfrak{c}$ for the annihilator of $\mathcal{O}/\mathcal{O}'$, call it as the *conductor* of $\mathcal{O}$ into $\mathcal{O}'$. It is a coherent sheaf of ideals on X' and its variety S' is a set of points on X'

which are not normal. Put $S = p^{-1}(S')$, so that p gives an isomorphism between $X\backslash S$ and $X'\backslash S'$, i.e. p is a birational morphism. For $Q \in X'$, $\mathfrak{c}_Q = \{f \in \mathcal{O}'_Q : fg \in \mathcal{O}'_Q \; \forall g \in \mathcal{O}_Q\}$, which is also the largest ideal of $\mathcal{O}'_Q$ that is also an ideal of $\mathcal{O}_Q$. We have a chain of inclusions

$$\mathcal{O}_Q \supset k + \mathfrak{r}_Q \supset \mathcal{O}'_Q \supset k + \mathfrak{c}_Q$$

where $\mathfrak{r}_Q$ is the radical of $\mathcal{O}_Q$, same as the set of $f \in \mathcal{O}_Q$ such that $f(P) = 0$ for all $P \mapsto Q$.

1.2 Case of an algebraic curve

In case of curves, a curve is normal if and only if nonsingular, so normalization corresponds to *desingularization* for curves. For a normalization $p : X \to X'$, the corresponding $S' \subset X'$ and $S = p^{-1}(S') \subset X$ are finite sets, and especially, S' is nothing but the set of singular points of X'. For all $Q \in X'$, $\delta_Q := \dim(\mathcal{O}_Q/\mathcal{O}'_Q)$ is finite, and positive if and only if $Q \in S'$. δ_Q is also an *analytic invariant*: it is preserved under taking completions: $\delta_Q = \dim(\widehat{\mathcal{O}_Q}/\widehat{\mathcal{O}'_Q})$. We call that two singular points are *analytically isomorphic* if they have same δ_Q.

We know that $\mathcal{O}_Q/\mathcal{O}'_Q$, $\mathcal{O}'_Q/\mathfrak{c}_Q$, and $\mathcal{O}_Q/\mathfrak{c}_Q$ are all finite dimensional, so $\mathfrak{c}_Q \supset \mathfrak{r}_Q^n$ for some $n > 0$ and we get

$$k + \mathfrak{r}_Q \supset \mathcal{O}'_Q \supset k + \mathfrak{r}_Q^n. \tag{1}$$

Question 1.1. For given X' and $Q \in X'$, how can we compute δ_Q?

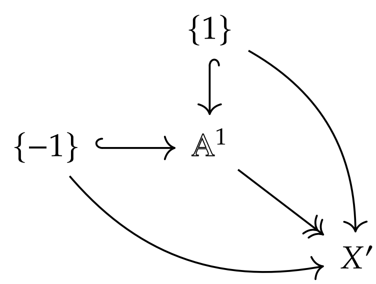
1.3 Construction of a singular curve from its normalization

We can also "denormalize" a nonsingular curve X, i.e. do opposite direction of above. More precisely, let X be an irreducible nonsingular curve and $S \subset X$ be a finite set. Let R be an equivalence relation on S, and define $S' := S/R$. Then Proposition 2 says that $X' := (X\backslash S) \cup S'$ with the canonical projection $p : X \to X'$ becomes an algebraic curve with singularities at S'. Intuitively, we are identifying points in S under the equivalence relation R to get a singular curve X' with singular points S'.

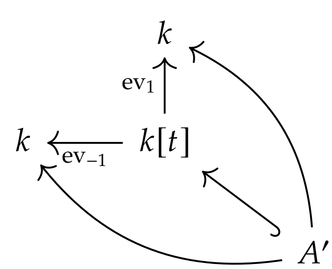
Here's a sketch of the "construction" of $\mathcal{O}'$, the structure sheaf of X'. Put $\mathcal{O}_Q := \cap_{P \to Q} \mathcal{O}_P$ (taking intersection) and $\mathfrak{r}_Q$ to be the radical of $\mathcal{O}_Q$. For each $Q \in S'$, choose $\mathcal{O}'_Q \subset \mathcal{O}_Q$ different from $\mathcal{O}_Q$ so that (1) is satisfied for some n. For $Q \in X'\backslash S'$, define $\mathcal{O}'_Q = \mathcal{O}_Q$. Then this will form a sheaf $\mathcal{O}'$.

To prove that $(X, \mathcal{O}_X)$ is a normalization of $(X', \mathcal{O}')$, it is enough to check when X is affine. Assume that A is a coordinate ring of X. Let $A' = \cap_{Q \in X'} \mathcal{O}'_Q \subset A$. For each $P \in X$, let $\mathfrak{a}_P = \{f \in A : f(P) = 0\} \subset A$ be the maximal ideal corresponds to P and $\mathfrak{r} = \cap_{P \in S} \mathfrak{a}_P$. By (1) there exists n such that $A' \supset k + \mathfrak{r}^n$, and one can show that A is an A'-module of finite type; so is integral over A'. Then A' is also a k-algebra of finite type and we have a corresponding affine variety Y, and X becomes a normalization of Y (A is an integral closure of A'). Now we can prove that Y is actually isomorphic to $(X', \mathcal{O}')$.

The above proof is constructive, i.e. it gives a coordinate ring A' of a corresponding X' when $X = \operatorname{Spec} A$ is affine. Here's a different approach that might be easier to understand, at least for me. Consider the case when $X = \mathbb{A}^1 = \operatorname{Spec} k[x]$ and $S = \{1, -1\}$, so that we are identifying two points on a line to get a singular curve. Then the resulting $X' = X/\{1 \sim -1\} = \operatorname{Spec} A'$ should correspond to a universal object to the following diagram



which corresponds to the diagram of rings



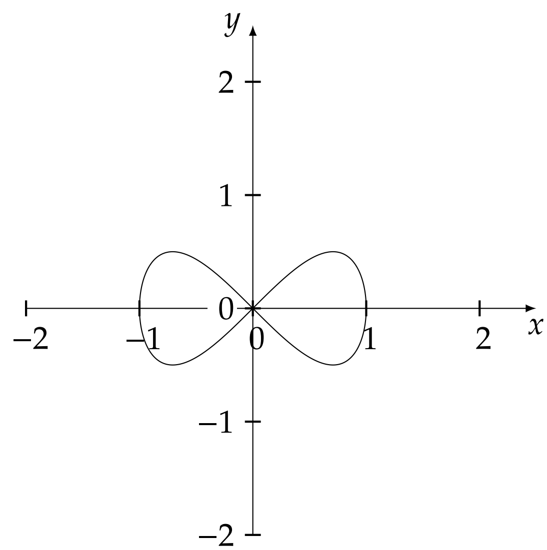
In other words, A' is a subring $A' = \{f \in k[t] : f(-1) = f(1)\}$. One can show that A' is generated by two elements $t^2 - 1$ and $t^3 - t$, i.e. $A' = k[t^2 - 1, t^3 - t]$, and A' is not integrally closed since $t = \frac{t^3 - t}{t^2 - 1}$ is in its fractional field, and integral over A' ($t^2 = (t^2 - 1) + 1$), but $x \notin A'$. We also know that $A'[t] = k[t] = A$, so A is an integral closure of A'. Now, one can prove that $A' \simeq k[x, y]/(y^2 - x^3 - x^2)$ via $(t^2 - 1, t^3 - t) \leftrightarrow (x, y)$.

Here's a slightly more complicated example: $X = k[s, t]/(s^2 + t^2 - 1)$ and $S = \{(0, -1), (0, 1)\}$, i.e. identifying two points on a circle. Let $B' = \{f \in$

$k[s,t] : f(0,-1) = f(0,1)\}$ and $A' = B'/(s^2+t^2-1)$. One can show that $B' = k[t^2-1, t(t^2-1), s, st]$, hence

$$A' = k[-s^2, -s^2t, s, st]/(s^2+t^2-1) = k[s, st]/(s^2+t^2-1).$$

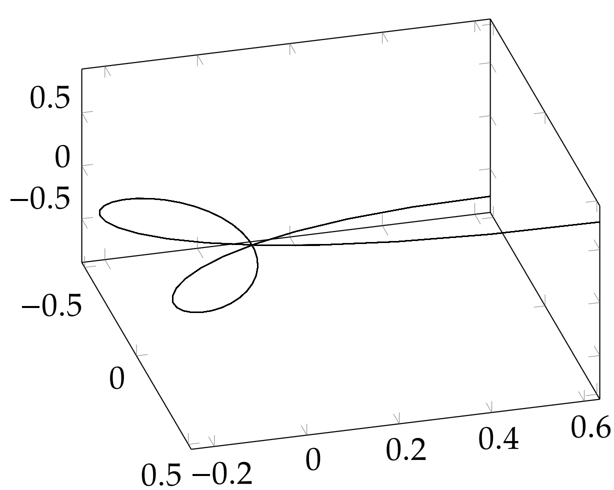
and this is isomorphic to $k[x,y]/(y^2 - x^2(1-x^2))$ under the map $(s, st) \leftrightarrow (x, y)$. This gives an equation $y^2 = x^2(1-x^2)$ of X'.



Another examples is when we identify three distinct points. Consider $X = \mathbb{A}^1$ and $S = \{-1, 0, 1\}$. By the above arguments, $X' = X/S = \operatorname{Spec} A'$ where $A' = \{f \in k[t] : f(-1) = f(0) = f(1)\}$. One can show that

$$A' = k[t(t^2-1), t^2(t^2-1), t^3(t^2-1)] \simeq k[x,y,z]/(y^2 - xz, x^4 - y(y^2-x^2)),$$

i.e. X' is an intersection of two surfaces $y^2 = xz$ and $x^4 = y(y^2-x^2)$ in $\mathbb{A}^3$.



1.4 Singular curve defined by a modulus

Using the previous construction, we can define a (singular) curve $X_{\mathfrak{m}}$ from a modulus $\mathfrak{m}$ of X. Here we assume $\deg(\mathfrak{m}) \geq 2$ (otherwise $X_{\mathfrak{m}} = X$). Let S be the support of $\mathfrak{m}$. Take $S' = \{Q\}$ to be a single point and $X' := (X \backslash S) \cup S'$ (i.e. merge every singular points of X into a single point). Put

$$\begin{aligned} \mathfrak{c}_Q &:= \{f \in \mathcal{O}_Q : f \equiv 0 \pmod{\mathfrak{m}}\} \\ \mathcal{O}'_Q &:= k + \mathfrak{c}_Q \end{aligned}$$

Then we can apply the previous construction to get a singular curve $X_{\mathfrak{m}}$ with a unique singular point Q, where

$$\delta_Q = \dim(\mathcal{O}_Q/\mathcal{O}'_Q) = \dim(\mathcal{O}_Q/(k + \mathfrak{c}_Q)) = \dim(\mathcal{O}_Q/\mathfrak{c}_Q) - 1 = \deg(\mathfrak{m}) - 1.$$

For example, when $\mathfrak{m} = 2P$ for some $P \in X$, the corresponding $X_{\mathfrak{m}}$ with $p : X \to X_{\mathfrak{m}}$ has an *ordinary cusp* at $Q = p(P)$, i.e. analytically isomorphic to $y^2 - x^3$ at Q. When $\mathfrak{m} = P_1 + P_2$ with $P_1 \neq P_2$, it identifies two different points on X, and the resulting curve $X_{\mathfrak{m}}$ is analytically isomorphic to $xy = 0$ at the singular point Q (i.e. Q is a *node*).

Question 1.2. For a given irreducible nonsingular curve X (in terms of certain set of equations) and a modulus $\mathfrak{m}$, how to compute $X_{\mathfrak{m}}$, i.e. how to find a corresponding equation of $X_{\mathfrak{m}}$? Can we do this at least for $X = \mathbb{P}^1$ or $\mathbb{A}^1$?

2 Riemann–Roch theorems

2.1 Notations

Now assume X (and so X') are complete curves, so also projective. Let g be the genus of X and put

$$\begin{aligned} \delta &= \sum_{Q \in S'} \delta_Q \\ \pi &= \delta + g. \end{aligned}$$

For a divisor D on X prime to S, we have a sheaf $\mathcal{L}(D)$ on X associated to it. Under the birational map $X \to X'$, we can define a sheaf $\mathcal{L}'(D)$ on X' via

$$\mathcal{L}'(D)_Q := \begin{cases} \mathcal{L}(D)_Q & Q \notin S' \\ \mathcal{O}'_Q & Q \in S'. \end{cases}$$

Also, we define

$$L'(D) = \mathrm{H}^0(X', \mathcal{L}'(D)), \quad I'(D) = \mathrm{H}^1(X', \mathcal{L}'(D)),$$
$$l'(D) = \dim L'(D), \quad i'(D) = \dim I'(D)$$

as in the nonsingular case. Especially, when $X' = X_{\mathfrak{m}}$, we denote above as $L_{\mathfrak{m}}(D), I_{\mathfrak{m}}(D), l_{\mathfrak{m}}(D), i_{\mathfrak{m}}(D)$.

2.2 The Riemann–Roch theorem (first form)

The Riemann–Roch theorem for singular curves has a same form as nonsingular case, but just replace $l(D), i(D), g$ with $l'(D), i'(D), \pi$: for any divisor D prime to S,

$$l'(D) - i'(D) = \deg(D) + 1 - \pi.$$

We can prove it by using the same argument as in Chapter 2 for nonsingular curves: by considering a cohomology sequence, on can reduce it to the case of $D = 0$, i.e. $\chi(X', \mathcal{O}') = 1 - \pi$. Using $0 \to \mathcal{O}' \to \mathcal{O} \to \mathcal{O}/\mathcal{O}' \to 0$ and $\chi(X', \mathcal{O}/\mathcal{O}') = \dim \mathrm{H}^0(X', \mathcal{O}/\mathcal{O}') = \delta$ ($\mathcal{O}/\mathcal{O}'$ is supported on S'), it reduces to prove that $\chi(X', \mathcal{O}) = 1 - g$. This follows from the fact that $\mathrm{H}^q(X, \mathcal{O}_X) = \mathrm{H}^q(X', \mathcal{O})$ for all $q \geq 0$: this holds essentially because $X \to X'$ is a finite map.

2.3 Application to the computation of the genus of an algebraic curve

By considering $D = 0$, we get $\pi = i'(0) = \dim \mathrm{H}^1(X', \mathcal{O}')$. As a corollary, we can compute *arithmetic genus* $p_a(X')$ of a curve: recall that it is defined as

$$p_a(X') = 1 - \chi(X') = 1 - \dim \mathrm{H}^0(X', \mathcal{O}') + \dim \mathrm{H}^1(X', \mathcal{O}')$$

Since X' is connected, $\dim \mathrm{H}^0(X', \mathcal{O}') = 1$ and we get

$$p_a(X') = \pi = \delta + g.$$

Note that $p_a(X')$ depends on the base field k; see also Stacks project (but again, we assume k is algebraically closed). This is useful since $p_a(X')$ and δ are supposed to be easy to compute, and we can use them to compute g. For example, if X' is a complete intersection of $r - 1$ hypersurfaces of degre $a_1, \ldots, a_{r-1}$ in $\mathbb{P}^r$ we get

$$\pi = \frac{1}{2} a_1 \cdots a_{r-1} \left(\sum_{i=1}^{r-1} a_i - r - 1 \right) + 1$$

and when $r = 2$, we get a *Plücker formula*

$$g = \frac{1}{2} d(d-3) + 1 - \delta.$$

2.4 Genus of a curve on a surface

Let V be a projective non-singular surface and $X' \subset V$ be a curve in it. Then we have a general formula for $p_a(X')$ in terms of a certain intersection number:

$$p_a(X') = 1 + \frac{1}{2} X' \cdot (X' + K)$$

where K is a canonical divisor of V, i.e. divisor of a non-zero differential form of degree 2.

3 Differentials on a singular curve

3.1 Regular differentials on X'

A differential ω on X is said to be *regular at* $Q \in S'$ if $\sum_{P \to Q} \mathrm{Res}_P(f\omega) = 0$ for all $f \in \mathcal{O}'_Q$. We denote $\underline{\Omega}'_Q$ for the set of regular differentials, which is a $\mathcal{O}'_Q$-submodule of $D_k(X)$. If we put

$$\underline{\Omega}_Q := \bigcup_{P \to Q} \underline{\Omega}_P,$$

then $\underline{\Omega}_Q \subset \underline{\Omega}'_Q$ and we have a duality between $\mathcal{O}_Q / \mathcal{O}'_Q$ and $\underline{\Omega}'_Q / \underline{\Omega}_Q$, given by the pairing $(f, \omega) \mapsto \sum_{P \to Q} \mathrm{Res}(f\omega)$ above. When $X' = X_{\mathfrak{m}}$ with $\mathfrak{m} = \sum_P n_P P$, ω is regular on X' if and only if

$$\sum_{P \in S} \mathrm{Res}_P(\omega) = 0 \quad \text{and} \quad v_P(\omega) \geq -n_P \; \forall P \in S.$$

For general case, regular differentials are characterized as follows: ω is everywhere regular on X' if and only if $\mathrm{Tr}_g(\omega) = 0$ for every rational function g on X which is not a p-th power (if $\mathrm{char}(k) = p$) and which belongs to all $\mathcal{O}'_Q$, $Q \in S'$.

3.2 Duality theorem

Serre duality also extends to singular curves. For a divisor D prime to S, associate a sheaf $\underline{\Omega}'(D)$ by

$$\underline{\Omega}'(D)_Q = \begin{cases} \underline{\Omega}'_Q & Q \in S' \\ \underline{\Omega}(D)_Q & Q \notin S'. \end{cases}$$

We also put $\Omega'(D) = \mathrm{H}^0(X', \underline{\Omega}'(D))$. Then $\omega \in \Omega'(D)$ if and only if it is regular at all $Q \in S'$ and also $v_P(\omega) \geq v_P(D)$ for all $P \notin S$. Then we have $\Omega'(D) \simeq I'(D)^* =$

$\mathrm{H}^1(X', \mathcal{L}'(D))^*$, and we get $i'(D) = \dim \Omega'(D)$. Especially, $i'(0) = \dim \Omega'(0) = \pi$, i.e. the dimension of everywhere regular differential forms on X' is π. Proof uses the adèlic language in Chapter 2.

3.3 The equality $n_Q = 2\delta_Q$

For $Q \in S'$, we can identify the conductor ideal $\mathfrak{c}_Q$ with a divisor $\sum_{P \to Q} n_P P$ such that $\mathfrak{c}_Q = \{f \in \mathcal{O}_Q : (f) \geq \sum_{P \to Q} n_P P\}$. By using the duality between $\mathcal{O}_Q/\mathcal{O}'_Q$ and $\underline{\Omega}'_Q/\underline{\Omega}_Q$, one can see that they have the same annihilators and $f \in \mathfrak{c}_Q$ iff $v_P(f\omega) \geq 0$ for all $P \to Q$ and $\omega \in \underline{\Omega}'_Q$. In other words, $n_P = \sup_{\omega \in \underline{\Omega}'_Q}(-v_P(\omega))$. Now put $n_Q := \sum_{P \to Q} n_P$ be the degree of the divisor $\mathfrak{c}_Q$. Then we have an inequality between n_Q and δ_Q:

$$\delta_Q + 1 \leq n_Q \leq 2\delta_Q$$

for all $Q \in S'$, and $n_Q = 2\delta_Q$ if and only if $\underline{\Omega}'_Q$ is a free $\mathcal{O}'_Q$-module of rank 1. When this holds for all $Q \in S'$, $\underline{\Omega}'$ is locally free and corresponds to a class of divisors of K' on X', where $K = K' - \mathfrak{c}$. Taking deg gives $\deg(K') = 2\pi - 2$, and $\underline{\Omega}'(D) \simeq \mathcal{L}'(K' - D)$, which gives a definitive form of the Riemann–Roch theorem

$$l'(D) - l'(K' - D) = \deg(D) + 1 - \pi.$$

For the previous examples of $\mathfrak{m} = P_1 + P_2$ or $\mathfrak{m} = 2P$, the theorem shows that $\delta_Q = 1$ for $Q \in X_{\mathfrak{m}}$, since $n_Q = 2$ and $1 = \frac{n_Q}{2} \leq \delta_Q \leq n_Q - 1 = 1$ for both cases.

3.4 Complements

Note that $n_Q = 2\delta_Q$ holds when

- X' is a complete intersection in a projective space,
- X' is embedded in a nonsingular surface V.