Automorphic induction from GL_1/K to $\mathrm{GL}_2/\mathbb{Q}$

Seewoo Lee

February 15, 2022

In this note, we show that for any given Hecke character ξ of a quadratic field $K/\mathbb{Q}$, there exists a GL_2 automorphic form over $\mathbb{Q}$. This is a part of conjectured *automorphic induction*: for any degree r field extension K/F, an automorphic representation of GL_n over K induces an automorphic representation GL_{rn} over F. Essentially, there are two kinds of automorphic forms of GL_2: (holomorphic) modular forms and Maass forms. Here we give proofs for both cases, where the first case is proved by Hecke and the second case is proved by Maass.

1 Converse theorems of L-functions

For any given modular form (or even an automorphic form over $\mathrm{GL}(n)$) $f(z) = \sum_{n\geq 0} a_n e^{2\pi i n z}$, we can define a L-function

$$L_f(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$$

which has a meromorphic continuation (with simple poles at $s = 0, k$), functional equation and bounded on any vertical strip. Hecke's converse theorem gives a converse of this. He proved that if a certain L-function satisfies the above properties, then it is a L-function comes from a modular form. More precisely:

Theorem 1. *Let k be a positive even number. Suppose $f(z) = \sum_{n\geq 0} a_n e^{2\pi i n z}$ with $a_n = O(n^\alpha)$ for some positive constant $\alpha > 0$. Then $f \in \mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$, i.e. f is a modular form on $\mathrm{SL}_2(\mathbb{Z})$ of weight k if and only if the function*

$$\Lambda_f(s) = (2\pi)^{-s}\Gamma(s)\sum_{n\geq 1}\frac{a_n}{n^s}$$

can be analytically continued over the whole s-plane,

$$\Lambda_f(s) + \frac{a_0}{s} + \frac{i^k a_0}{k-s}$$

is entire and bounded in vertical strips, and satisfies the functional equation

$$\Lambda_f(s) = i^k \Lambda_f(k-s).$$

Proof uses the following theorem with $q = 1$, which is also proved by Hecke.

Theorem 2 (Hecke)**.** *Suppose f and g are given by the Fourier series*

$$f(z) = \sum_{n \geq 0} a_n e^{2\pi i n z}, \quad g(z) = \sum_{n \geq 0} b_n e^{2\pi i n z}$$

with coefficients a_n, b_n bounded by $O(n^\alpha)$ for $n \geq 1$ where $\alpha > 0$ is a constant. Let

$$L_f(s) = \sum_{n \geq 1} \frac{a_n}{n^s}, \quad L_g(s) = \sum_{n \geq 1} \frac{b_n}{n^s}$$

be corresponding L-functions and

$$\Lambda_f(s) = \left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma(s) L_f(s), \quad \Lambda_g(s) = \left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma(s) L_g(s)$$

be completed L-functions, where q is a given positive number. Put $\omega = \left(\begin{smallmatrix} & -1 \\ q & \end{smallmatrix}\right)$ and $(f|_\omega)(z) = (\sqrt{q}z)^{-k} f(-1/qz)$, where k is a given positive integer. TFAE:

1. *$g = f|_\omega$.*
2. *Both $\Lambda_f(s)$ and $\Lambda_g(s)$ have meromorphic continuation over the whole s-plane,*

$$\Lambda_f(s) + \frac{a_0}{s} + \frac{b_0 i^k}{k - s}$$
$$\Lambda_g(s) + \frac{b_0}{s} + \frac{a_0 i^{-k}}{k - s}$$

   are entire and bounded on vertical strips, and they satisfy

$$\Lambda_f(s) = i^k \Lambda_g(k - s).$$

Proof of this theorem uses the Mellin transform, the Phragmén-Lindelöf convexity principle and Striling's estimate for the gamma function. (See [6].)

To generalize this result for arbitrary levels, we need more functional equations, which are given by *twisting* the original L-functions by characters. Weil proved the following converse theorem:

Theorem 3 (Weil)**.** *Let $k \geq 1$ be an integer and χ a character mod $q \geq 1$. Let f, g be a function on $\mathcal{H}$ defined by*

$$f(z) = \sum_{n \geq 0} a_n e^{2\pi i n z}, \quad g(z) = \sum_{n \geq 0} b_n e^{2\pi i n z}$$

with coefficients $\{a_n\}, \{b_n\}$ bounded by $O(n^\alpha)$ for all $n \geq 1$, where $\alpha > 0$ is a constant. Suppose $\Lambda_f(s), \Lambda_g(s)$ is defined by

$$\Lambda_f(s) = \left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma(s) L_f(s), \quad \Lambda_g(s) = \left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma(s) L_g(s)$$

satisfies the following: both $\Lambda_f(s), \Lambda_g(s)$ have meromorphic continuation over the whole s-plane,

$$\Lambda_f(s) + \frac{a_0}{s} + \frac{b_0 i^k}{k-s}, \quad \Lambda_g(s) + \frac{b_0}{s} + \frac{a_0 i^{-k}}{k-s}$$

are entire and bounded on vertical strips, and they satisfy

$$\Lambda_f(s) = i^k \Lambda_g(k-s).$$

Let $\mathcal{R}$ be a set of prime numbers coprime to q which meets every primitive residue class, i.e. for any $c > 0$ and any a with $(a, c) = 1$ there exists $r \in \mathcal{R}$ such that $r \equiv a \pmod{c}$. Suppose for any primitive character ψ of conductor $r \in \mathcal{R}$ the functions

$$\Lambda_f(s, \psi) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L_f(s, \psi) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) \sum_{n \geq 1} \frac{a(n)\psi(n)}{n^s}$$

$$\Lambda_g(s, \psi) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L_g(s, \psi) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) \sum_{n \geq 1} \frac{b(n)\psi(n)}{n^s}$$

with $N = qr^2$ are entire, bounded in vertical strips and satisfy the functional equation

$$\Lambda_f(s, \psi) = i^k w(\psi) \Lambda_g(k-s, \overline{\psi})$$

with $w(\psi) = \chi(r)\psi(q)\tau(\psi)^2 r^{-1}$, where $\tau(\psi) = \sum_{u \,(\mathrm{mod}\, r)} \psi(u) e^{2\pi i u/r}$ is a Gauss sum. Then $f \in \mathcal{M}_k(\Gamma_0(q), \chi)$, i.e. modular form of weight k on $\Gamma_0(q)$ with a character χ, and $f \in \mathcal{M}_k(\Gamma_0(q), \overline{\chi})$. Also, $g = f|_\omega$ where $\omega = \omega_q = \left(\begin{smallmatrix} & -1 \\ q & \end{smallmatrix}\right)$. Moreover, f, g are cusp forms if $L_f(s)$ or $L_g(s)$ converges absolutely on some line $\Re s = \sigma$ with $0 < \sigma < k$.

2 Hecke L-function

For a number field K, we can define a Dedekind zeta function

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{(N\mathfrak{a})^s}$$

where $\mathfrak{a}$ ranges through nonzero ideals of $\mathcal{O}_K$, and $N\mathfrak{a} := [\mathcal{O}_K : \mathfrak{a}]$ denotes the absolute norm of $\mathfrak{a}$. This is a generalization of a zeta function ($\zeta_{\mathbb{Q}}(s)$ coincides with the original Riemann zeta function) and also satisfies some similar properties - has an analytic continuation and a functional equation. We can also *twist* it by some characters as we can do for zeta functions (which is called a Dirichlet L-function). Such character is called a *Hecke character*, which is a homomorphism from I, the group of fractional ideals, to $\mathbb{C}$.

To define such L-function, we first define Hecke character. Let I be the group of nonzero fractional ideals of K and $P \leq I$ be the subgroup of principal ideals. Then the group $\mathrm{Cl}(K) = I/P$ is called the class group of K, which is known to be finite. For any integral ideal $\mathfrak{m} \subset \mathcal{O}_K$, we define

$$I_{\mathfrak{m}} = \{\mathfrak{a} \in I \,:\, (\mathfrak{a}, \mathfrak{m}) = 1\}$$
$$P_{\mathfrak{m}} = \{(a) \in P \,:\, a \equiv 1 \pmod{\mathfrak{m}}\}$$

Then the group $\mathrm{Cl}_{\mathfrak{m}}(K) := I_{\mathfrak{m}}/P_{\mathfrak{m}}$ is also finite, which is called the ray class group.

Now define a homomorphism $\xi_\infty : K^\times \to S^1$ by the product

$$\xi_\infty(a) = \prod_\sigma \left(\frac{a^\sigma}{|a^\sigma|}\right)^{u_\sigma} |a^\sigma|^{iv_\sigma}$$

where σ ranges through all the embeddings $\sigma : K \to \mathbb{C}$ and the numbers u_σ, v_σ are given with the following restrictions:

$$\begin{array}{ll} u_\sigma = 0, 1 & \text{if } \sigma \text{ is real} \\ u_\sigma \in \mathbb{Z} & \text{if } \sigma \text{ is complex} \\ v_\sigma \in \mathbb{R} & \text{such that } \sum_\sigma v_\sigma = 0 \end{array}$$

Then we can find a smallest integral ideal $\mathfrak{m}$ such that the group

$$U_{\mathfrak{m}} = \{\eta \in U_K = \mathcal{O}_K^\times \,:\, \eta \equiv 1 \pmod{\mathfrak{m}}\}$$

is in $\ker \xi_\infty$. We call such $\mathfrak{m}$ as a modulus of ξ_∞. By definition, ξ_∞ can be regarded as a function on $P_{\mathfrak{m}}$.

A group homomorphism $\xi : I_{\mathfrak{m}} \to S^1$ is said to be a character to modulus $\mathfrak{m}$ if $\xi|_{P_{\mathfrak{m}}} = \xi_\infty$. Note that if ξ is a character to modulus $\mathfrak{m}$, then it is a character to modulus $\mathfrak{n}$ for any $\mathfrak{n} \subseteq \mathfrak{m}$. We call that ξ is primitive if it is not induced by other character with a smaller modulus. We can extend such homomorphism to $\xi : I \to \mathbb{C}$ by setting $\xi(\mathfrak{a}) = 0$ for $(\mathfrak{a}, \mathfrak{m}) \neq 1$. Such map is called *Hecke character* or *Grossencharacter* of K. For any given Hecke character ξ, we can define Hecke L-function

$$L(s, \xi) = \sum_{0 \neq \mathfrak{a} \subseteq \mathcal{O}_K} \frac{\xi(\mathfrak{a})}{(N\mathfrak{a})^s}.$$

Like other L-functions, it also has the Euler product

$$L(s, \xi) = \prod_{\mathfrak{p}} (1 - \xi(\mathfrak{p})(N\mathfrak{p})^{-s})^{-1}.$$

Hecke proved that the L-function has an analytic countinuation and a functional equation.

Theorem 4 (Hecke)**.** *Let $\xi \pmod{m}$ be a primitive nontrivial Hecke character of a number field K. Put*

$$\Lambda(s, \xi) = (2^{r_1}(2\pi)^{-n}|D|N\mathfrak{m})^{\frac{s}{2}} \prod_\sigma \Gamma\left(\frac{1}{2}(|u_\sigma| + n_\sigma(s + iv_\sigma))\right) L(s, \xi)$$

where

$$n_\sigma = \begin{cases} 1 & \sigma \text{ is real} \\ 2 & \sigma \text{ is complex} \end{cases}.$$

The function $\Lambda(s,\xi)$ is entire and bounded in vertical strips, and it satisfies the functional equation

$$\Lambda(s,\xi) = w(\xi)\Lambda(1-s,\overline{\xi})$$

where

$$w(\xi) = i^{-u}W(\xi)(N\mathfrak{m})^{-1/2}, \quad u = \sum_\sigma u_\sigma.$$

Here $W(\xi)$ is a Gauss sum

$$W(\xi) = \frac{\xi_\infty(b)}{\xi(\mathfrak{t})} \sum_{a \in \mathfrak{t}/\mathfrak{tm}} \xi_{\mathfrak{m}}(a) e^{2\pi i (\mathrm{Tr}(a/b))z},$$

where $\mathfrak{t}$ is an integral ideal prime to $\mathfrak{m}$ such that $\mathfrak{t}\mathfrak{d}\mathfrak{m}$ is principal, say $\mathfrak{t}\mathfrak{d}\mathfrak{m} = (b)$ with $b \in \mathcal{O}_K$. The sum does not depend on the choice of $\mathfrak{t}$ and b. $\xi_{\mathfrak{m}}$ is a character of $K^\times$ given by

$$\xi_{\mathfrak{m}}(a) = \frac{\xi((a))}{\xi_\infty(a)}, \quad a \in K^\times$$

which is periodic of period $\mathfrak{m}$.

3 Modular form associated with imaginary quadratic fields

Now we prove that we can attach a modular form to a given Hecke character of an imaginary quadratic field. Let $K = \mathbb{Q}(\sqrt{D})$ be an imaginary quadratic field with discriminant $D < 0$. Let $\chi_D(n) = \left(\frac{n}{D}\right)$ be a Kronecker symbol, so that $\chi_D(-1) = 1, -1$ if K is real or imaginary, and

$$\chi_D(p) = \begin{cases} 0 & p \text{ ramifies in } K \\ 1 & p \text{ splits in } K \\ -1 & p \text{ inerts in } K \end{cases}$$

Theorem 5. *Let $\xi \pmod{\mathfrak{m}}$ be a Hecke character of K such that*

$$\xi((a)) = \left(\frac{a}{|a|}\right)^u \quad \text{if } a \equiv 1 \pmod{\mathfrak{m}}$$

where u is a non-negative integer. Then

$$f(z) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \xi(\mathfrak{a})(N\mathfrak{a})^{\frac{u}{2}} e^{2\pi i (N\mathfrak{a})z} \in \mathcal{M}_k(\Gamma_0(N), \chi)$$

where $k = u+1$, $N = |D| \cdot N\mathfrak{m}$ and $\chi \pmod{N}$ is the Dirichlet character given by

$$\chi(n) = \chi_D(n)\xi((n)), \quad n \in \mathbb{Z}.$$

Proof. Here we only give a proof when ξ is primitive. Consider

$$g(z) = C \sum_{\mathfrak{a}} \overline{\xi}(\mathfrak{a})(N\mathfrak{a})^{\frac{u}{2}} e^{2\pi i (N\mathfrak{a}) z}$$

where $C = i^{-2u-1} W(\xi)(N\mathfrak{m})^{-1/2}$. By definition, we have $L_f(s) = L(s - \frac{u}{2}, \xi)$ and $L_g(s) = CL(s - \frac{u}{2}, \overline{\xi})$. Now replace s with $s - \frac{u}{2}$ in the functional equation

$$\Lambda(s, \xi) = i^{-u} W(\xi)(N\mathfrak{m})^{-1/2} \Lambda(1 - s, \overline{\xi})$$

and we get the functional equation

$$\Lambda_f(s) = i^k \Lambda_g(k - s)$$

where

$$\Lambda_f(s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L\left(s - \frac{u}{2}, \xi\right), \quad \Lambda_g(s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) CL\left(s - \frac{u}{2}, \overline{\xi}\right).$$

Hence by the converse theorem it follows that

$$g = f|_{\omega_N}, \quad \omega_N = \begin{pmatrix} & -1 \\ N & \end{pmatrix}.$$

Next let $\psi \pmod p$ be a primitive Dirichlet character of conductor $p \nmid N$. To show that f is in $\mathcal{M}_k(\Gamma_0(N), \chi)$, we apply the Weil's converse theorem to the completed L-functions

$$\begin{aligned} \Lambda_f(s, \psi) &= \left(\frac{p\sqrt{N}}{2\pi}\right)^s \Gamma(s) L\left(s - \frac{u}{2}, \xi \cdot \psi \circ N\right) \\ \Lambda_g(s, \psi) &= \left(\frac{p\sqrt{N}}{2\pi}\right)^s \Gamma(s) CL\left(s - \frac{u}{2}, \overline{\xi} \cdot \psi \circ N\right), \end{aligned}$$

which satisfy the functional equation

$$\Lambda_f(s, \psi) = i^k w(\psi) \Lambda_g(k - s, \overline{\psi})$$

with $w(\psi) = \chi(p)\psi(N)\tau(\psi)^2 p^{-1}$, which follows from the functional equation of Hecke L-function again. Here we use the identity $W(\psi \circ N) = \chi_D(p)\psi(|D|)\tau(\psi)^2$. (This identity follows from the L-function factorization

$$L(s, \psi \circ N) = L(s, \psi) L(s, \psi\chi_D)$$

and comparing the functional equations of both sides.) $\square$

4 Maass form associated with real quadratic fields

By the same argument, we can prove that there exists a Maass form of weight 0 with an eigenvalue 1/4 corresponds to a Hecke character of a real quadratic field. First, we prove that there exists a corresponding weight 1 modular form.

Theorem 6. *Let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field with discriminant $D > 0$ and $\xi \pmod{\mathfrak{m}}$ a Hecke character such that*

$$\xi((a)) = \frac{a}{|a|} \quad \text{if } a \equiv 1 \pmod{\mathfrak{m}}$$

or

$$\xi((a)) = \frac{a'}{|a'|} \quad \text{if } a \equiv 1 \pmod{\mathfrak{m}}$$

where a' denotes the conjugate over $\mathbb{Q}$. Then

$$f(z) = \sum_{\mathfrak{a}} \xi(\mathfrak{a}) e^{2\pi i (N\mathfrak{a})z} \in \mathcal{S}_1(\Gamma_0(N), \chi)$$

where $N = D \cdot N\mathfrak{m}$ and the character $\chi \pmod{N}$ is defined in the previous theorem.

Proof. Define

$$g(z) = C \sum_{\mathfrak{a}} \overline{\xi}(\mathfrak{a}) e^{2\pi (N\mathfrak{a})z}$$

where $C = -W(\xi)(N\mathfrak{m})^{-1/2}$. By definition, we have $L_f(s) = L(s, \xi)$ and $L_g(s) = CL(s, \overline{\xi})$. The completed L-function $\Lambda(s, \xi)$ is given by

$$\begin{aligned}\Lambda(s, \xi) &= (2^2(2\pi)^{-2}N)^{s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)L(s, \xi) \\ &= 2\sqrt{\pi}\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(s, \xi).\end{aligned}$$

Here we use the duplication formula of the Gamma function

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = 2^{1-s}\sqrt{\pi}\Gamma(s).$$

Then from the functional equation

$$\Lambda(s, \xi) = i^{-1}W(\xi)(N\mathfrak{m})^{-1/2}\Lambda(1-s, \xi)$$

we get the functional equation

$$\Lambda_f(s) = i\Lambda_g(1-s)$$

where

$$\Lambda_f(s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(s, \xi), \quad \Lambda_g(s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)CL(s, \overline{\xi}).$$

Hence by the converse theorem it follows that

$$g = f|_{\omega_N}, \quad \omega_N = \begin{pmatrix} & -1 \\ N & \end{pmatrix}.$$

Next let $\psi \pmod p$ be a primitive Dirichlet character of conductor $p \nmid N$. To show that f is in $\mathcal{S}_1(\Gamma_0(N), \chi)$, we apply the Weil's converse theorem to the completed L-functions

$$\Lambda_f(s, \psi) = \left(\frac{p\sqrt{N}}{2\pi}\right)^s \Gamma(s) L\left(s, \xi \cdot \psi \circ N\right)$$
$$\Lambda_g(s, \psi) = \left(\frac{p\sqrt{N}}{2\pi}\right)^s \Gamma(s) C L\left(s, \overline{\xi} \cdot \psi \circ N\right),$$

which satisfy the functional equation

$$\Lambda_f(s, \psi) = i w(\psi) \Lambda_g(1 - s, \overline{\psi})$$

with $w(\psi) = \chi(p)\psi(N)\tau(\psi)^2 p^{-1}$, which follows from the functional equation of Hecke L-function again. □

Corollary 1. *Let K, ξ, and χ be as in the previous theorem. Then*

$$u(z) = \sum_{\mathfrak{a}} \xi(\mathfrak{a}) y^{1/2} e^{2\pi i (N\mathfrak{a}) z}$$

is a Maass cusp form of weight 1 *with an eigenvalue* $1/4$ *and a character* χ *on* $\Gamma_0(N)$.

Proof. Use the fact that if $f(z)$ is a modular form of weight k, then $z \mapsto y^{k/2} f(z)$ is a Maass form of weight k with an eigenvalue $\frac{k}{2}\left(1 - \frac{k}{2}\right)$. $u(z)$ is a cusp form since $f(z)$ is. □

Note that $e^{-y} = W_{1,0}(y)$, where $W_{k,ir}(y)$ is the Whittaker function, which is the exponentially-decaying solution of the differential equation

$$y^2 \frac{d^2 W}{dy^2} + y \frac{dW}{dy} - (y^2 - ky - r^2) W = 0.$$

One can check that $\mathcal{W}_{k,ir}(y) := \sqrt{y} W_{k,ir}(2\pi y) e^{2\pi i x}$ is an eigenfunction of the weight k Laplacian with an eigenvalue $\frac{1}{4} + r^2$, i.e. $\Delta_k \mathcal{W}_{k,ir} = (\frac{1}{4} + r^2)\mathcal{W}_{k,ir}$. We can rewrite the function $u(z)$ as

$$u(z) = \sum_{n \geq 1} c_n n^{-1/2} \mathcal{W}_{1,0}(nz)$$

where $c_n = \sum_{N\mathfrak{a} = n} \xi(\mathfrak{a})$.

As you can see, every weight 1 Maass form with an eigenvalue $1/4$ comes from a weight 1 modular form, and it is known that there exists a Galois representation associated with such modular form. (This is a theorem of Deligne-Serre, see [3].) It is conjectured that we can attach a Galois representation to any Maass form with an eigenvalue $1/4$ (even for weight $k \neq 1$), and this is still open. Cohen construct an explicit example of Maass form of weight 0 with an eigenvalue $1/4$ which has a corresponding Galois representation (See [2]).

5 Known cases of automorphic induction

So we just proved the automorphic induction from GL_1/K to $\mathrm{GL}_2/\mathbb{Q}$ when $K/\mathbb{Q}$ is a quadratic extension. In general, automorphic induction is an open problem, but there are some known cases (see [10]):

- Local fields (Henniart-Herb, [5])
- Cyclic Galois extension of prime degree (Arthur-Clozel, [1])
- Non-normal cubic extension (Jacquet-Piatetski-Shapiro-Shalika, [8] and [9])
- Non-normal extensions with solvable Galois closure for certain Hecke characters (Harris, [4])
- Non-normal quintic extension with non-solvable closure (Kim, [7])

References

[1] J. Arthur, L. Clozel, *Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula*, Annals of Mathematics Studies, Princeton University Press, 1989.

[2] H. Cohen, *q-identities for Maass waveforms*, Invent. Math. 91, no. 3, 409-422, 1988.

[3] P. Deligne and J. P. Serre. *Formes modulaires de poids 1.* Ann. Sci. École Norm. Sup. 7, no. 4, Elsevier, 1974.

[4] M. Harris, *The local Langlands conjecture for* GL(n) *over a p-adic field,* $n < p$, Invent. Math. 134, no. 1, 177-210, 1998.

[5] G. Henniart, R. Herb, *Automorphic induction for* GL(n) *(over nonarchmedean fields)*, Duke Math J. 78, no. 1, 131-192, 1995.

[6] H. Iwaniec, *Topics in Classical Automorphic Forms*, American Mathematical Soc., 1997.

[7] H. Kim, *An example of non-normal quintic automorphic induction and modularity of symmetric powers of cusp forms of icosahedral type*, 2003.

[8] H. Jacquet, I. Piatetski-Shapiro, J. Shalika, *Automorphic forms on* GL(3) *I*, Ann. of Math. 109, no. 1, 169-212, 1979.

[9] H. Jacquet, I. Piatetski-Shapiro, J. Shalika, *Automorphic forms on* GL(3) *II*, Ann. of Math. 109, no. 2. 213-258, 1979.

[10] Laie, *The status of automorphic induction*, question on mathoverflow, `https://mathoverflow.net/questions/152263/the-status-of-automorphic-induction`.