# Strange series and Irrationality measure

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Consider the following problems:

1. Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2 n}$$

2. Does the following sequence bounded?

$$a_n = \frac{1}{|n^2 \sin n|}$$

Two problems look alike and both aren't easy. In fact, both problems are open. First one is called Flint-Hills series and we don't know the answer yet. To introduce one approach to attack these problems, we introduce the concept of irrationality measure.

**Definition 1.** *Let $x$ be a real number and let $R = R(x)$ be the set of positive real numbers $\mu$ for which*

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^\mu}$$

*has at most finitely many solutions $p/q$ for $p$ and $q$ integers. Then the irrationality measure of $x$ is defined as $\mu(x) := \inf_{\mu \in R(x)} \mu$.*

If $x$ can be approximated by rational numbers well, then $\mu(x)$ will be small. In fact, we have the following proposition:

**Proposition 1.** *For $x \in \mathbb{Q}$, $\mu(x) = 1$.*

*Proof.* Let $x = \frac{r}{s}$ with $\gcd(r, s) = 1$. First, assume that $\mu > 1$. Then

$$0 < \left| \frac{r}{s} - \frac{p}{q} \right| < \frac{1}{q^\mu} \Rightarrow 0 < |qr - ps| < \frac{s}{q^{\mu-1}}$$

and $qr - ps$ is a nonzero integer, so $|qr - ps| \geq 1$. Since $\mu > 1$, $\frac{s}{q^{\mu-1}} < 1$ for large $q$, so the equation only has finitely many solutions. For $\mu = 1 - \epsilon$ with small $\epsilon > 0$, there exists infinitely many $(p, q)$ with $|ps - qr| = 1$ (since $\gcd(r, s) = 1$), and then $0 < |qr - ps| = 1 < sq^\epsilon$ for sufficiently large $q$. □

For $x \notin \mathbb{Q}$, we have Dirichlet's theorem:

**Theorem 1** (Dirichlet)**.** *For any $x \in \mathbb{R}\backslash\mathbb{Q}$, the inequality*

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}$$

*has infinitely many solutions. Hence, for $x \in \mathbb{R}\backslash\mathbb{Q}$, we have $\mu(x) \geq 2$.*

*Proof.* Proof follows from pigeonhole principle. Assume that $x \notin \mathbb{Q}$. We will show the following: for any $N > 0$, there exists integers $p, q$ with $1 \leq q \leq N$ s.t.

$$|qx - p| < \frac{1}{N}.$$

Consider the following set: $\{0, \{x\}, \{2x\}, \{3x\}, \ldots, \{Nx\}\}$, where $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$ is a fractional part of $\alpha$. By pigeonhole principle, there exists $i > j$ s.t. $|\{ix\} - \{jx\}| < \frac{1}{N}$. If we put $q = i - j$ and $p = \lfloor ix \rfloor - \lfloor jx \rfloor$, then we get $|qx - p| < \frac{1}{N}$. From this, we have

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{Nq} \leq \frac{1}{q^2}.$$

□

We have a natural question: for which $x$, $\mu(x) = 2$? Roth's theorem shows that $\mu(x) = 2$ for algebraic numbers with degree $> 1$.

**Theorem 2** (Roth)**.** *Let $x \notin \mathbb{Q}$ be an algebraic number and $\epsilon > 0$. Then*

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\epsilon}}$$

*has at most finitely many solutions $p/q$ for $p$ and $q$ integers. Hence, we have $\mu(x) = 2$ for $x \in \overline{\mathbb{Q}}\backslash\mathbb{Q}$.*

This is a very hard theorem and Roth got Fields medal by proving this. Until now, we don't know much about irrationality measure of transcendental numbers. However, we can express $\mu(x)$ in terms of continued fraction of $x$.

**Theorem 3** (Sondow)**.** *Let $x = [a_0, a_1, a_2, \ldots]$ be a simple continued fraction of $x$ and $p_n/q_n$ be $n$-th convergent. Then*

$$\mu(x) = 1 + \limsup_{n \to \infty} \frac{\ln q_{n+1}}{\ln q_n} = 2 + \lim \sup_{n \to \infty} \frac{\ln a_{n+1}}{\ln q_n}$$

To prove this, we need a lemma:

**Lemma 1** (Legendre)**.** *For integers $p, q$ with*

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2},$$

*$p/q$ is a convergent of the continued fraction expansion of $x$.*

*Proof.* Define $\lambda_n$ by

$$\left| x - \frac{p_n}{q_n} \right| = q_n^{-\lambda_n}.$$

One can show that

$$\frac{1}{2q_n q_{n+1}} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}},$$

so we have

$$\frac{1}{2q_n q_{n+1}} < \frac{1}{q_n^{\lambda_n}} < \frac{1}{q_n q_{n+1}}.$$

By taking logarithm, Lemma 1 implies that

$$\mu(x) = \limsup_{n\to\infty} \lambda_n = 1 + \limsup_{n\to\infty} \frac{\ln q_{n+1}}{\ln q_n}.$$

□

As a corollary, one can show that every quadratic irrational number has irrationality measure 2, since their continued fractions are periodic, so $\log q_n \sim n \log \beta$ for some $\beta \in \overline{\mathbb{Q}}$. Also, simple continued fraction of $e$ is

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, \dots]$$

and we can prove that $\mu(e) = 2$ from this.

Now we go back to the original problems. Both problems are related to irrationality measure of $\pi$. Let $\mu = \mu(\pi)$ and $\alpha, \beta > 0$. We are going to estimate the sequence

$$A_n^{(\alpha,\beta)} = n^\alpha |\sin n|^\beta.$$

**Theorem 4.** *For $\alpha, \beta > 0$ with $\frac{\alpha}{\beta} < \mu - 1$, $\inf_{n\geq 1} A_n^{(\alpha,\beta)} = 0$.*

*Proof.* Assume that $\beta(\mu - 1) > \alpha$. Choose $\epsilon > 0$ such that $\beta(\mu - \epsilon - 1) > \alpha$.

$$\left| \pi - \frac{p}{q} \right| < \frac{1}{q^{\mu-\epsilon}}$$

has infinitely many solutions.

$$|\sin p| = |\sin(p - q\pi)| = \sin |p - q\pi| < |p - q\pi| < \frac{1}{q^{\mu-\epsilon-1}}$$

$$A_p^{(\alpha,\beta)} = p^\alpha |\sin p|^\beta < \frac{p^\alpha}{q^{\beta(\mu-\epsilon-1)}} < \frac{(4q)^\alpha}{q^{\beta(\mu-\epsilon-1)}} = \frac{4^\alpha}{q^{\beta(\mu-\epsilon-1)-\alpha}}$$

and so $\inf_{n\geq 1} A_n^{(\alpha,\beta)} = 0$. □

Now assume that Flint-Hills series converges. Then we have $\lim_{n\to\infty} 1/A_n^{(3,2)} = 0$, hence $\inf_{n\geq 1} A_n^{(3,2)} > 0$ and $\mu \leq 1+3/2 = 5/2$. Similarly, if we can prove that $\{a_n\}$ is bounded, then $\inf_{n\geq 1} 1/a_n = \inf_{n\geq 1} 1/A_n^{(2,1)} > 0$, so $\mu \leq 1 + 2/1 = 3$. Note that the current record of the upperbound of $\mu(\pi)$ is 7.6063 which is proved by Salikov (see [1]).

There's another interesting series from MO:

$$\sum_{n=1}^{\infty} \frac{|\sin(n)|^n}{n}.$$

This also seems that we have to deal with irrationality measure of $\pi$. Actually, this series converges and proved by Terrence Tao (see [3]). He actually proved the stronger result

$$\sum_{n=1}^{\infty} \frac{|\sin(n)|^n}{n^{1-\frac{1}{2(\mu-1+\epsilon)}}} < \infty$$

where $\mu = \mu(\pi)$. He use the fact that $\mu(\pi)$ is finite.

## References


[1] V.Salikhov. *On the Irrationality Measure of* $\pi$, Usp. Mat. Nauk 63, 163-164, 2008

[2] J. Sondow, *Irrationality Measures, Irrationality Bases, and a Theorem of Jarnik*, Proceedings of Journées Arithmétiques, Graz 2003 in the Journal du Theorie des Nombres Bordeaux.

[3] T. Tao, answer to the MO question *Is the series* $\sum_n |\sin n|^n/n$ *convergent?*, mathoverflow, `https://mathoverflow.net/questions/282259/is-the-series-sum-n-sin-nn-n-convergent`