How automorphic forms and elliptic curves fly?

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Abstract

This is an expository note on *murmurations*, which was initially discovered by He, Lee, Oliver, and Pozdnyakov for elliptic curves. We focus on the cases where the mumuration density is computed (under GRH), including the work of Zubrilina, Lee–Oliver–Pozdnyakov, and Sawin–Sutherland.

1 Introduction

Murmuration is a recently discovered phenomenon in number theory, referring to striking oscillatory patterns that appear when averaging (normalized) Frobenius traces over families of L-functions, especially as a function of the ratio between a prime p and a parameter like conductor or height. The phenomenon was first observed for elliptic curves by He, Lee, Oliver, and Pozdnyakov [13] who found that the average of $a_p(E)$ over isogeny classes of elliptic curves with fixed rank and conductor in a given range exhibits a universal oscillation pattern depending only on the rank. Sutherland [33] further clarified that the pattern depends on p/N (with N the conductor), and that similar behavior appears when averaging with root number weights, or over dyadic intervals.

Subsequent works have established and computed murmuration densities in other settings, such as Dirichlet characters [18], modular forms [37, 22, 21, 3], Hecke characters of imaginary quadratic fields [36], and Maass forms [4]. In some cases, the density function (distribution) can be written explicitly, while in others (notably elliptic curves) it remains mysterious or highly intricate. Recent progress by Sawin and Sutherland [25] has established a version of murmuration for elliptic curves ordered by height, with local averaging. Most of the works uses trace formulas in their proofs.

A general framework for murmuration was proposed by Sarnak [23], relating the phenomenon to the Katz–Sarnak philosophy of low-lying zeros and symmetry types of families of L-functions [16, 17]. The existence and form of murmuration densities are closely tied to the conductor dimension of the family and the need for local averaging.

This note surveys the main results and techniques in the study of murmuration, focusing on the cases of elliptic curves, Dirichlet characters, modular forms, and related families. We also discuss the general formulation and its connection to random matrix theory.

Not all existing works on murmuration are included in this note. For example, Cowan [7] studied the murmuration using ratio conjecture, which also fit into the general framework of Sarnak [23].

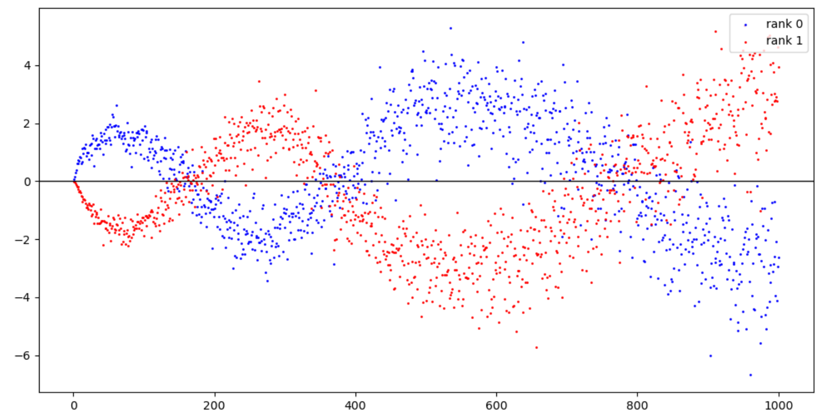


Figure 1: Murmuration of elliptic curves with conductor in $[7500, 1000]$ *and rank* $r = 0$ *(blue) and* $r = 1$ *(red) [13].*

1.1 Plots

Most of the plots are reproduced by the author. Some of them have fewer data points or smaller conductor ranges than the original ones due to computational limitations. Readers can find the code in the GitHub repository: `https://github.com/seewoo5/murmuration`.

2 Murmuration of Elliptic Curves

2.1 He–Lee–Oliver–Pozdnyakov's Murmuration

Murmuration of elliptic curves refers to the following average of Frobenius traces. Fix a nonnegative integer r and $N_1 < N_2$. Let $\mathcal{E}_r[N_1, N_2]$ be the set of isogenous classes of elliptic curves $E_{/\mathbb{Q}}$ with conductor $N(E) \in [N_1, N_2]$ and rank r. For a fixed prime p, we consider the following average:

$$\mathbb{E}_{E \in \mathcal{E}_r[N_1,N_2]}[a_p(E)] = \frac{\sum_{E \in \mathcal{E}_r[N_1,N_2]} a_p(E)}{\sum_{E \in \mathcal{E}_r[N_1,N_2]} 1} \tag{1}$$

as a function of p. What He, Lee, Oliver, and Pozdnyakov [13] observed is that this yields a surprising oscillation pattern, as in Figure 1. In particular, it appears to have the same oscillation pattern for different conductor ranges, where the pattern seems to depend only on the rank r.

2.2 Sutherland's observation

When the paper is uploaded on arXiv, Sutherland was interested in the work and sent a letter to Rubinstein and Sarnak [33] asking if murmuration can be explained by the known results, with further experiments. In his letter, he observed that one really needs to view the murmuration density as a function of p/N rather than p for a fixed N. He found that, for different dyadic intervals of the form $[2^k, 2^{k+1})$, the murmuration patterns look the same (and become clearer as k increases), even if the averages consider completely different sets of elliptic curves (Figure 2). Also, instead of considering each rank separately, it seems better to consider all ranks together, where we weight $a_p(E)$ by the root number $\epsilon(E)$ of E. One can separate into two groups depending on the parity of the rank. So the major open question is to *compute* the density function, i.e., to find a function $M : (0, \infty) \to \mathbb{R}$ such that

$$\mathbb{E}_{E \in \mathcal{E}_r[N,2N]}[a_p(E)\epsilon(E)] = M\left(\frac{p}{N}\right) + \text{error} \tag{2}$$

where the error term goes to zero as $N \to \infty$. More generally, one can fix $0 < C_1 < C_2$ and consider the interval $[C_1N, C_2N]$.

Sutherland also observed that the murmuration disappears when elliptic curves are ordered by other measures, such as naive height, discriminant, or j-invariants, although further local averaging gives murmuration for naive heights (see Section 5). This shows that the murmuration is a phenomenon that is sensitive to the ordering of elliptic curves.

2.3 What is the role of Machine Learning?

Although there seems to be no machine learning involved in the previous discussions, I will make a brief comment on the relation between machine learning and murmuration, as I found that existing literature is often misleading in distinguishing the machine learning part from the murmuration part. I have read a few articles on the internet which basically say that "AI found new mathematics," which is false.

One of the main motivations of the papers [12, 13] is to study elliptic curves via machine learning. In particular, they were interested in predicting the rank of elliptic curves (which is widely known to be hard to compute in general) by means of machine learning, where the coefficients $a_p(E)$ of Hasse–Weil L-functions are used as features. Surprisingly, they found that a simple logistic regression model can already distinguish between rank 0 and 1 elliptic curves with high accuracy of $> 90\%$. Along these lines, they (more precisely, He, Lee, and Oliver) were curious about what was actually going on, and Pozdnyakov (who was an undergraduate student of Lee at that time) figured out the murmuration pattern. This somehow gives an explanation for the high accuracy of the model, since the murmuration patterns for rank 0 and 1 elliptic curves are noticeably different. But the correct way to say it is that the machine learning experiments *motivated* them to study what the models were doing, which is essentially the work of humans, not the ML models. You can find more of the story in the Quanta Magazine article [6].

2.4 Sato–Tate conjecture, Plancherel density conjecture and Murmuration

One should not confuse murmuration with the (vertical) Sato–Tate conjecture or Plancherel density conjecture, which I will explain here. The original (i.e., *horizontal*) Sato–Tate conjecture is about the distribution of $a_p(E)$ for a *fixed* $E_{/\mathbb{Q}}$ *and varying* p. The Hasse–Weil bound says that $|a_p(E)| \leq 2\sqrt{p}$, and the conjecture predicts that for a non-CM elliptic curve E, the distribution of $a_p(E)$ is semicircular with radius $2\sqrt{p}$, i.e., the density function is $\frac{1}{2\pi}\sqrt{4-x^2}\mathrm{d}x$ for the normalized traces $a_p(E)/\sqrt{p}$. Equivalently, if we write $a_p(E) = 2\sqrt{p}\cos\theta_p$ for $\theta_p \in [0, \pi]$, then θ_p follows the distribution $\frac{2}{\pi}\sin^2\theta \mathrm{d}\theta$. The distributions for CM elliptic curves are different, and we also expect that the Frobenius traces for abelian varieties of higher dimension will follow certain distributions, which are conjecturally the pushforward of the Haar measure of a certain compact Lie group, called the *Sato–Tate group*. See [35] for more about the Sato–Tate conjecture and recent progress on it.

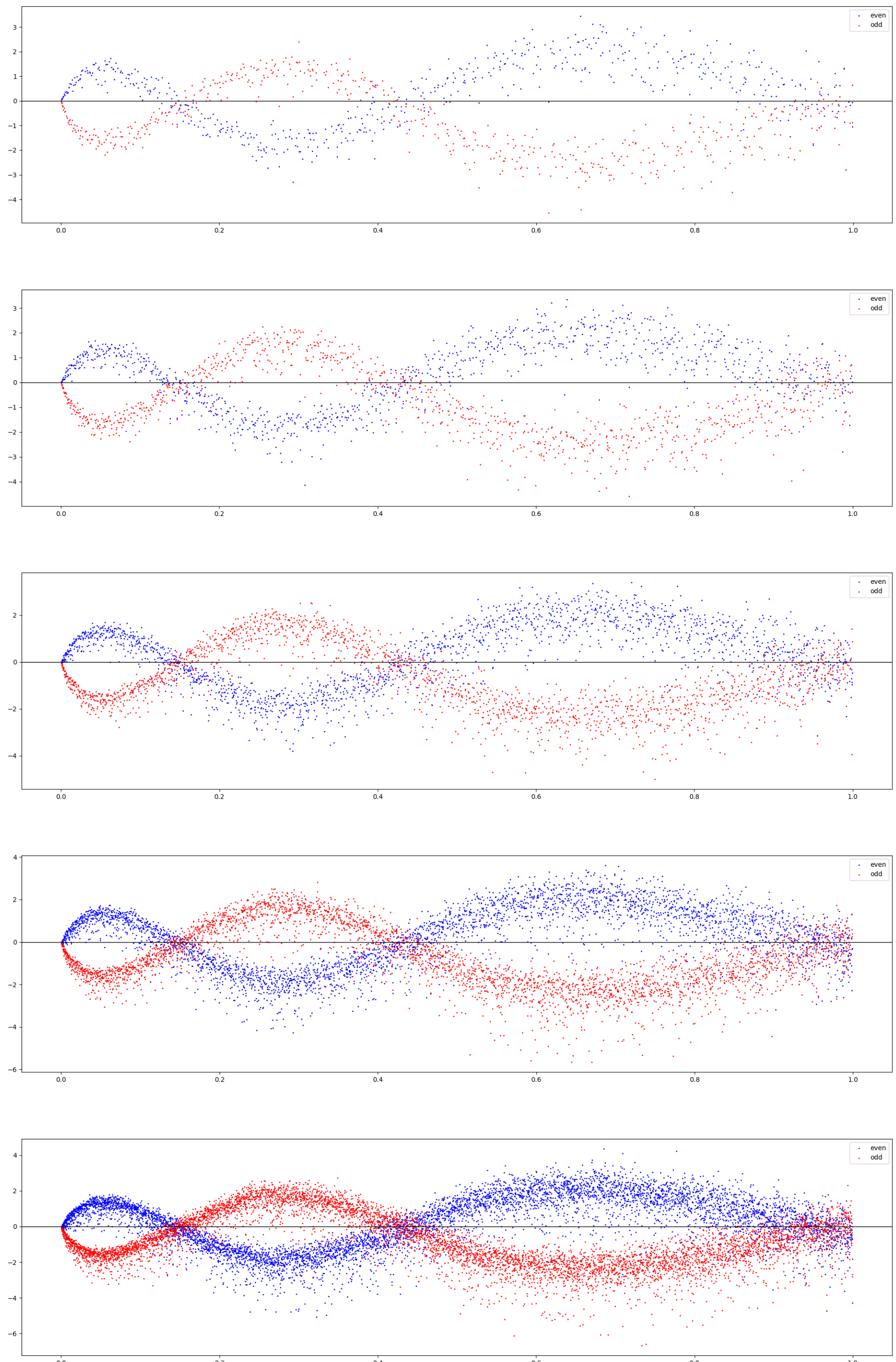


Figure 2: Murmuration of non-CM elliptic curves with conductor in $[2^k, 2^{k+1})$ *and primes* $p < 2^k$ *for* $k = 12, \ldots, 16$. *Blue (resp. red) curves correspond to* $\epsilon(E) = +1$ *(resp.* -1*) elliptic curves.*

The *vertical* Sato–Tate conjecture fixes p and varies E over $\mathbb{F}_p$ instead, where there are only finitely many isomorphism classes of E over $\mathbb{F}_p$. Birch [2] proved that the distribution converges to the above semicircular distribution as $p \to \infty$. This is different from the murmuration for two reasons: vertical Sato–Tate considers the elliptic curves over $\mathbb{F}_p$, and there's no conductor involved in vertical Sato–Tate.

Probably, a closer situation would be fixing a prime p and varying E over $\mathbb{Q}$ with conductors in a certain range. Although we couldn't find a work in this direction, the most relevant case would be the corresponding case of modular forms by Serre [27]. In particular, he proved that: if a sequence of pairs of weight and level (k_i, N_i) satisfies $2 \mid k_i$, $k_i + N_i \to \infty$, and $p \nmid N_i$, then the distribution of the normalized Hecke eigenvalues $a_p/p^{\frac{k-1}{2}} \in [-2, 2]$ converges to the Plancherel measure

$$\mu_p = \frac{p+1}{2\pi} \frac{\sqrt{4-x^2}}{(p^{\frac{1}{2}} + p^{-\frac{1}{2}})^2 - x^2} \mathrm{d}x$$

which is usually dubbed as a Plancherel density theorem (See also [29] and [30] for generalizations). Hence one can fix weight $k = 2$ and vary the level $N \to \infty$ with $p \nmid N$. However, this includes many more modular forms than the ones corresponding to elliptic curves. Also, we want both p and N to grow at similar rates in murmuration, so Serre's result does not directly apply to our situation.

3 Murmuration of Dirichlet Series

Although the original murmuration density for elliptic curves is still unknown, there are a few works where murmuration exists and is even computed (under GRH). Historically, the first such example is the work of Zubrilina on modular forms [37], but we will start with the simplest case of Dirichlet characters. Lee, Oliver, and Pozdnyakov computed the murmuration density for Dirichlet characters [18][1]. For complex characters, the corresponding murmuration densities are given by the following theorem.

Theorem 3.1 (Lee–Oliver–Pozdnyakov [18, Theorem 1.1])**.** Let $\mathcal{D}_+(N)$ (resp. $\mathcal{D}_-(N)$) denote the set of primitive even (resp. odd) Dirichlet characters modulo N. For $x \in \mathbb{R}_{>0}$, let $\lceil x \rceil^{\mathfrak{p}}$ be the smallest prime $\geq x$. For $c > 1$, $\delta > 0$, and $y > 0$, define

$$P_\pm(y, X, c) := \frac{\log X}{X} \sum_{\substack{N \in [X, cX] \\ N \text{ prime}}} \sum_{\chi \in \mathcal{D}_\pm(N)} \frac{\chi(\lceil yX \rceil^{\mathfrak{p}})}{\tau(\chi)}, \tag{3}$$

$$P_\pm(y, X, \delta) := \frac{\log X}{X^\gamma} \sum_{\substack{N \in [X, X+X^\gamma] \\ N \text{ prime}}} \sum_{\chi \in \mathcal{D}_\pm(N)} \frac{\chi(\lceil yX \rceil^{\mathfrak{p}})}{\tau(\chi)}. \tag{4}$$

Then

$$\lim_{X \to \infty} P_\pm(y, X, c) = \begin{cases} \int_1^c \cos\left(\frac{2\pi y}{x}\right) \mathrm{d}x & \text{if } +, \\ -i \int_1^c \sin\left(\frac{2\pi y}{x}\right) \mathrm{d}x & \text{if } -, \end{cases} \tag{5}$$

and assuming RH, if $\frac{1}{2} < \gamma < 1$, we have

$$\lim_{X \to \infty} P_\pm(y, X, \gamma) = \begin{cases} \cos(2\pi y) & \text{if } +, \\ -i \sin(2\pi y) & \text{if } -. \end{cases} \tag{6}$$

[1] These can be thought of as automorphic forms on GL_1 over $\mathbb{Q}$.

See Figure 3 for the plot of the above murmuration densities. As you can see, there are two versions of murmurations: the *long interval* $[X, cX]$ and the *short interval* $[X, X + X^{\delta}]$. Note that one needs to assume RH to get the short interval version, to guarantee the existence of primes in short intervals. The summand $\chi(p)/\tau(\chi)$ is the p-th Fourier coefficient of $\overline{\chi}$ when expanded in terms of additive characters: we have [14, eq. (3.12)]

$$\overline{\chi}(a) = \frac{1}{\tau(\chi)} \sum_{b \pmod N} \chi(b) \exp\left(\frac{2\pi i a b}{N}\right)$$

when $\tau(\chi) \neq 0$, which justifies the normalization (Note that $\mathcal{D}_{\pm}(N)$ is invariant under complex conjugation). Also, the above averages only consider prime moduli, though the authors also studied the case of composite moduli in [18, Section 6.1].

The proof of Theorem 3.1 is much simpler than the case of modular forms (Section 4). By orthogonality of characters, we have

$$\exp\left(\frac{2\pi i a}{N}\right) = \cos\left(\frac{2\pi a}{N}\right) + i \sin\left(\frac{2\pi a}{N}\right) = \frac{1}{\phi(N)} \sum_{\chi \pmod N} \overline{\chi}(a)\tau(\chi)$$

and taking $a = p$ and $-p$ for a prime $p \nmid N$ gives (note that $\tau(\chi_0) = -1$)

$$\cos\left(\frac{2\pi p}{N}\right) = -\frac{1}{\phi(N)} + \frac{1}{\phi(N)} \sum_{\substack{\chi \pmod N \\ \chi \neq \chi_0,\, \chi(-1)=1}} \tau(\overline{\chi})\chi(p),$$

$$\sin\left(\frac{2\pi p}{N}\right) = -\frac{i}{\phi(N)} \sum_{\substack{\chi \pmod N \\ \chi(-1)=-1}} \tau(\overline{\chi})\chi(p).$$

Now, use $\tau(\chi)\tau(\overline{\chi}) = N\chi(-1)$ for $\chi \in \mathcal{D}_{\pm}(N)$ to get [18, Lemma 2.6]: for two distinct primes p and N,

$$\sum_{\chi \in \mathcal{D}_+(N)} \frac{\chi(p)}{\tau(\chi)} = \left(\frac{N-1}{N}\right) \cos\left(\frac{2\pi p}{N}\right) + \frac{1}{N},$$

$$\sum_{\chi \in \mathcal{D}_-(N)} \frac{\chi(p)}{\tau(\chi)} = -i \left(\frac{N-1}{N}\right) \sin\left(\frac{2\pi p}{N}\right).$$

Combined with the prime number theorem (which gives equidistribution results of primes in $[X, cX]$ normalized by X), we get (5). For short intervals, RH and the prime number theorem imply

$$\lim_{X\to\infty} \frac{\log X}{X^{\gamma}} \cdot \#\{p \in [yX, yX + X^{\gamma}]\} = 1,$$

for $\frac{1}{2} < \gamma < 1$, and this implies [18, Lemma 2.9]

$$\lim_{X\to\infty} \frac{\log X}{X^{\gamma}} \sum_{\substack{p \in [yX, yX+X^{\gamma}] \\ p \text{ prime}}} f\left(\frac{p}{X}\right) = f(y) \tag{7}$$

which proves (6). Note that Baker, Harman, and Pintz proved an unconditional result for the existence of a prime in a slightly longer short interval $[X, X + X^{0.525}]$, which means that Theorem 3.1 holds with $\gamma \geq 0.525$ unconditionally [1].

They also proved similar results for real Dirichlet characters, but the proof is more complicated. Let $\mathcal{G}$ be the set of odd square-free integers and let $\chi_d = \left(\frac{d}{\cdot}\right)$. For a compactly supported smooth function $\Phi \geq 0$ on $\mathbb{R}$, define

$$M_\Phi(y, X, \gamma) = \frac{\log X}{X^{1+\gamma}} \sum_{\substack{p \in [yX, yX + X^\gamma] \\ p \text{ prime}}} \sum_{d \in \mathcal{G}} \Phi\left(\frac{d}{X}\right) \chi_{8d}(p)\sqrt{p}. \tag{8}$$

Theorem 3.2 (Lee–Oliver–Pozdnyakov [18, Theorem 1.2])**.** Fix $y > 0$ and assume $\frac{3}{4} < \gamma < 1$. Assuming GRH, we have

$$M_\Phi(y) := \lim_{X\to\infty} M_\Phi(y, X, \gamma) = \frac{1}{2} \sum_{\substack{a\geq 1 \\ a \text{ odd}}} \frac{\mu(a)}{a^2} \sum_{m\geq 1} (-1)^m \widetilde{\Phi}\left(\frac{m^2}{2a^2 y}\right), \tag{9}$$

where

$$\widetilde{\Phi}(\xi) = \int_{-\infty}^{\infty} (\cos(2\pi\xi x) + \sin(2\pi\xi x))\Phi(x)\mathrm{d}x. \tag{10}$$

(The limit does not depend on the choice of γ.)

See Figure 4 for the corresponding plots when Φ_+ (resp. Φ_-) is supported on $(1, 2)$ (resp. $(-2, -1)$). For the proof, we can write $M_\Phi(y, X, \gamma)$ as

$$\begin{aligned} M_\Phi(y, X, \gamma) &= \frac{\log X}{X^{1+\gamma}} \sum_{\substack{p \in [yX, yX + X^\gamma] \\ p \text{ prime}}} \sum_{\substack{d\in\mathbb{Z} \\ d \text{ odd}}} \mu^2(d)\Phi\left(\frac{d}{X}\right) \chi_{8d}(p)\sqrt{p} \\ &= \frac{\log X}{X^{1+\gamma}} \sum_{\substack{p \in [yX, yX + X^\gamma] \\ p \text{ prime}}} \sum_{\substack{d\in\mathbb{Z} \\ d \text{ odd}}} \left(\sum_{\substack{a^2 \mid d \\ 0<a}} \mu(a)\right) \Phi\left(\frac{d}{X}\right) \chi_{8d}(p)\sqrt{p} \\ &= M_{\Phi,A}(y, X, \gamma) + R_{\Phi,A}(y, X, \gamma), \end{aligned}$$

where $\beta := \sup_{x\in\mathbb{R}}\{|x| : \Phi(x) > 0\}$, $0 < A \leq \sqrt{\beta X}$, and

$$M_{\Phi,A}(y, X, \gamma) := \frac{\log X}{X^{1+\gamma}} \sum_{\substack{p \in [yX, yX + X^\gamma] \\ p \text{ prime}}} \sum_{\substack{d\in\mathbb{Z} \\ d \text{ odd}}} \left(\sum_{\substack{a^2 \mid d \\ 0<a\leq A}} \mu(a)\right) \Phi\left(\frac{d}{X}\right) \chi_{8d}(p)\sqrt{p}, \tag{11}$$

$$R_{\Phi,A}(y, X, \gamma) := \frac{\log X}{X^{1+\gamma}} \sum_{\substack{p \in [yX, yX + X^\gamma] \\ p \text{ prime}}} \sum_{\substack{d\in\mathbb{Z} \\ d \text{ odd}}} \left(\sum_{\substack{a^2 \mid d \\ A<a}} \mu(a)\right) \Phi\left(\frac{d}{X}\right) \chi_{8d}(p)\sqrt{p}. \tag{12}$$

We will show that $R_{\Phi,A}(y, X, \gamma) \to 0$ and $M_{\Phi,A}(y, X, \gamma)$ converges to the right-hand side of (9). Note that A will not be a fixed constant, but rather grows together with X: in fact, the proof uses $0 < \epsilon < (\gamma - \frac{3}{4})/5$ and $A = X^{1+5\epsilon-\gamma}$. To estimate the sum of the characters, one uses the Pólya–Vinogradov inequality: for a non-principal character $\chi \pmod{q}$,

$$\left|\sum_{M<n\leq M+N} \chi(n)\right| \ll q^{\frac{1}{2}} \log q \tag{13}$$

Using this, by assuming GRH, one can show that the summation over primes in the short interval $[yX, yX + X^\gamma]$ is bounded by (see [11, eq. (5.1)])

$$\left| \sum_{\substack{p \in [yX, yX+X^\gamma] \\ p \text{ prime}}} \chi_d(p) \right| \ll (yX)^{\frac{1}{2}+\epsilon} \tag{14}$$

for non-principal χ_d and $\frac{1}{2} < \gamma < 1$. Combined with Abel's summation formula, (14) shows that the remainder term $R_{\Phi,A}(y, X, \gamma)$ vanishes as $X \to \infty$, where $\gamma > \frac{3}{4}$ is used in the computation. For the main term $M_{\Phi,A}(y, X, \gamma)$, the Poisson summation formula implies [18, Lemma 2.11]

$$\frac{1}{X} \sum_{\substack{d \in \mathbb{Z} \\ d \text{ odd}}} \left(\sum_{\substack{a^2 || d| \\ a \le A}} \mu(a) \right) \Phi\left(\frac{d}{X}\right) \left(\frac{d}{p}\right) \sqrt{p} = \frac{1}{2} \left(\frac{2}{p}\right) \sum_{\substack{0 < a \le A \\ (a,2p)=1}} \frac{\mu(a)}{a^2} \sum_{k \in \mathbb{Z}} (-1)^k \left(\frac{k}{p}\right) \widetilde{\Phi}\left(\frac{kX}{2a^2p}\right),$$

and applying it to (11) gives

$$M_{\Phi,A}(y, X, \gamma) = \frac{\log X}{X^\gamma} \sum_{\substack{p \in [yX, yX+X^\gamma] \\ p \text{ prime}}} \frac{1}{2} \sum_{\substack{0 < a \le A \\ (a,2p)=1}} \frac{\mu(a)}{a^2} \sum_{0 \ne k \in \mathbb{Z}} (-1)^k \left(\frac{k}{p}\right) \widetilde{\Phi}\left(\frac{kX}{2a^2p}\right)$$

for sufficiently large X. Since $0 < a \le A \ll X^{\frac{1}{2}} \ll p$, we have $(a, 2p) = 1$ if and only if $(a, 2) = 1$, and switching the order of summation gives

$$M_{\Phi,A}(y, X, \gamma) = \frac{1}{2} \sum_{\substack{(a,2)=1 \\ 0 < a \le A}} \frac{\mu(a)}{a^2} \sum_{0 \ne k \in \mathbb{Z}} (-1)^k \frac{\log X}{X^\gamma} \sum_{\substack{p \in [yX, yX+X^\gamma] \\ p \text{ prime}}} \left(\frac{k}{p}\right) \widetilde{\Phi}\left(\frac{kX}{2a^2p}\right).$$

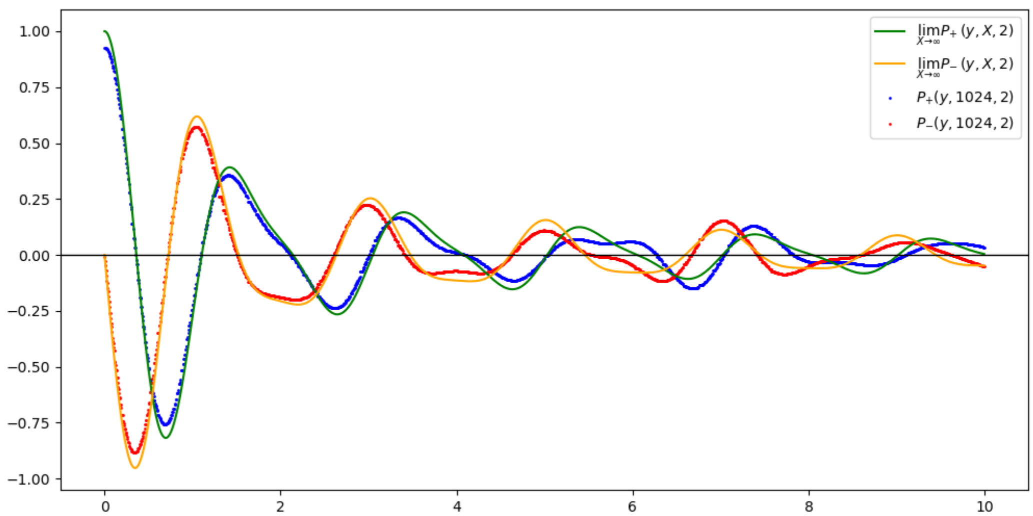
Now, using the Polya–Vinogradov inequality (14) again, one can show that the summation over non-square k exhibits cancellation and only the sum over square k contributes to the main term ($\gamma > \frac{3}{4}$ is used again):

$$\lim_{X \to \infty} M_{\Phi,A}(y, X, \gamma) = \lim_{X \to \infty} \frac{\log X}{X^\gamma} \sum_{\substack{p \in [yX, yX+X^\gamma] \\ p \text{ prime}}} \frac{1}{2} \sum_{\substack{(a,2)=1 \\ 0 < a \le A}} \frac{\mu(a)}{a^2} \sum_{\substack{m \ge 1 \\ (m,p)=1}} (-1)^m \widetilde{\Phi}\left(\frac{m^2X}{2a^2p}\right).$$

Finally, one can use the Poisson summation formula to show that the two inner sums on the right-hand side converge to a smooth function in $y = p/X$ as $X \to \infty$ (hence $A \to \infty$), and applying (7) gives the desired result. See [18, Section 6.2] for an analogous result for χ_d.

4 Murmuration of Modular Forms

The first murmuration density that was ever computed is for modular forms by Zubrilina [37]. For each fixed weight k, she computed the murmuration density of weight k cusp newforms of level $\Gamma_0(N)$, as $N \to \infty$. In this section, we briefly sketch the main ideas and results of her work.



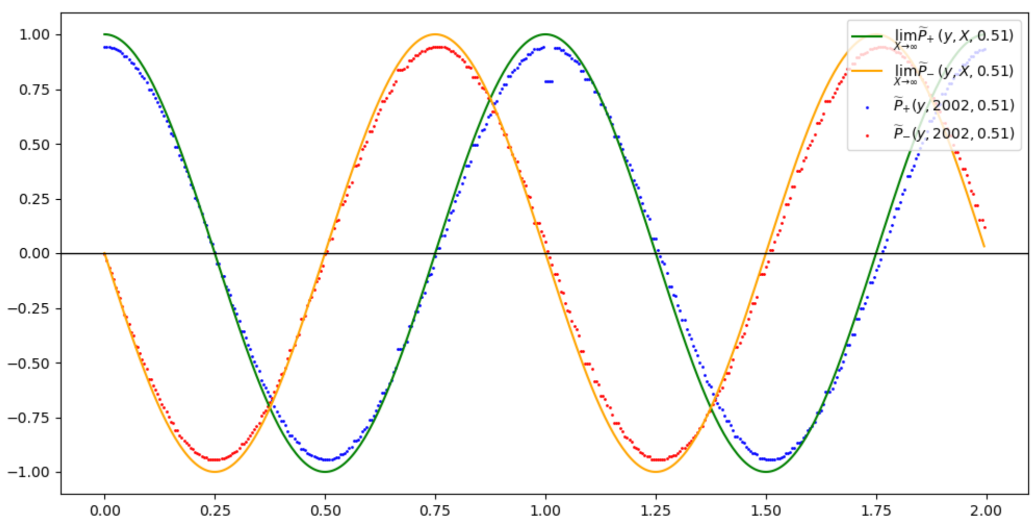


Figure 3: Murmuration of Dirichlet characters. The top figure presents $P_\pm(y, 2^{10}, 2)$ for $y \in [0, 10]$ with + in blue and (the imaginary part of) − in red. The bottom figure presents $\widetilde{P}_\pm(y, 2002, 0.51)$ for $y \in [0, 2]$ with + in blue and (the imaginary part of) − in red. The discontinuity of $\widetilde{P}_+(y, 2002, 0.51)$ at $y = 1$ corresponds to the term $p = N$ in (4).

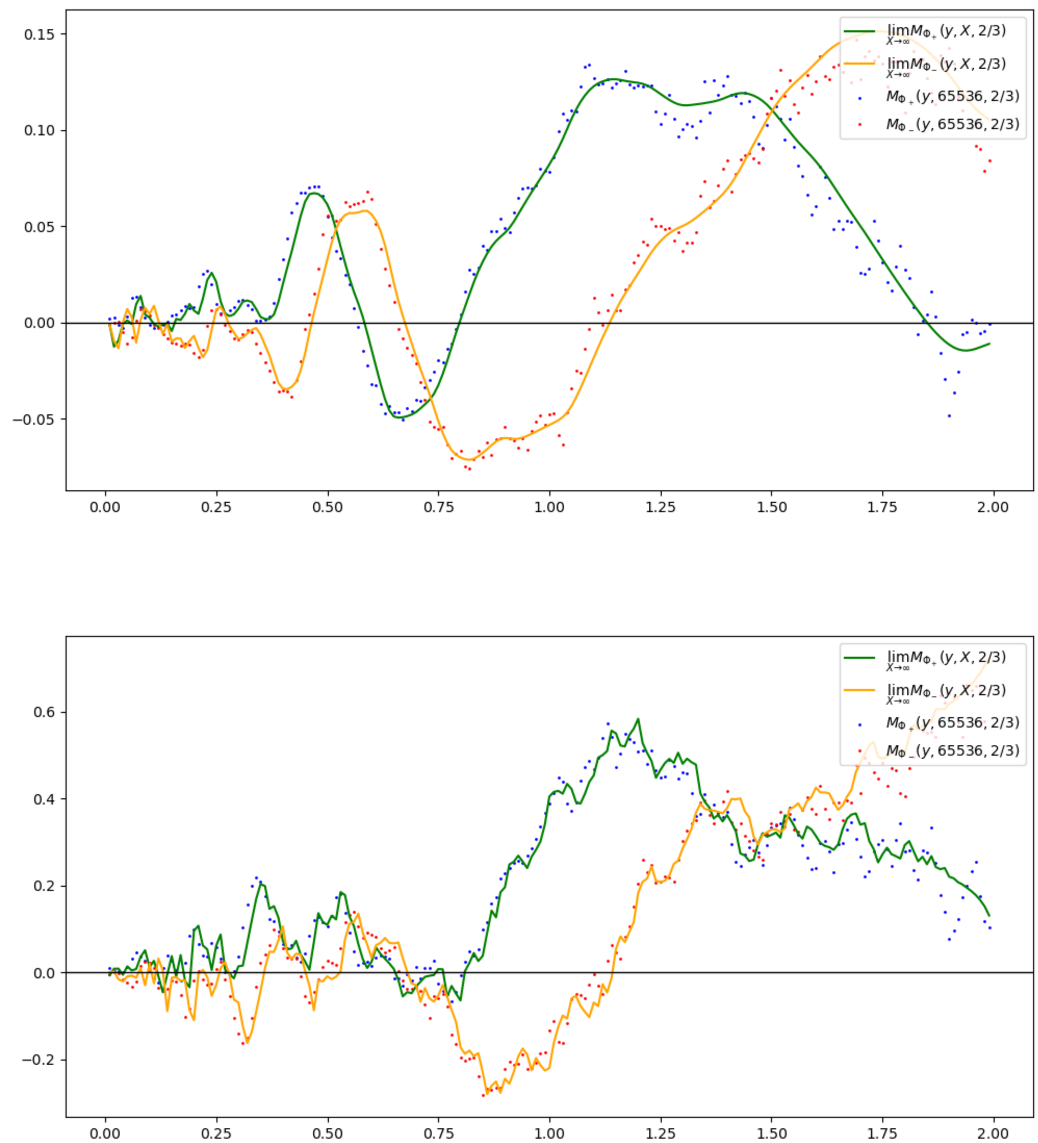


Figure 4: Murmuration of Kronecker characters χ_{8d}. The top figure presents $M_{\Phi_\pm}(y, 2^{16}, \frac{2}{3})$ for $y \in [0,2]$ with $\Phi_+(x) = \mathbb{1}_{(1,2)}(x)\exp(-1/(1-4(x-\frac{3}{2})^2))$ and $\Phi_-(x) = \mathbb{1}_{(-2,-1)}(x)\exp(-1/(1-4(x+\frac{3}{2})^2))$. The bottom figure presents $M_{\Phi_\pm}(y, 2^{16}, \frac{2}{3})$ for $y \in [0,2]$ with $\Phi_+(x) = \mathbb{1}_{(1,2)}(x)$ and $\Phi_-(x) = \mathbb{1}_{(-2,-1)}(x)$. Note that we use $\gamma = \frac{2}{3}$, even if the above proof only works when $\gamma > \frac{3}{4}$.

4.1 Statement

Before we state the result, we first define some notation. For each k and $N \geq 1$, let $H^{\mathrm{new}}(N,k)$ be the set of normalized Hecke cusp forms of weight k and level $\Gamma_0(N)$. For each $f \in H^{\mathrm{new}}(N,k)$, let $\epsilon(f) \in \{\pm 1\}$ be the root number of f, and $a_f(p)$ be the p-th Fourier coefficient of f and $\lambda_f(p) := a_f(p)/p^{\frac{k-1}{2}}$ be its analytic normalization. For $n \in \mathbb{Z}_{\geq 0}$, the *n-th Chebyshev polynomial of the second kind* is defined as

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

For each $r \in \mathbb{Z}_{\geq 1}$, define

$$\nu(r) := \prod_{p \mid r} \left(1 + \frac{p^2}{p^4 - 2p^2 - p + 1}\right)$$

Lastly, we define the constants α, β, γ as

$$\alpha := 2\pi \prod_p \frac{p^4 - 2p^2 - p + 1}{p^4 - 2p^2 + p}, \quad \beta := 2\pi \prod_p \frac{p^3 + p^2 - 1}{p(p^2 + p - 1)}, \quad \gamma := 12 \prod_p \frac{p(p+1)}{p^2 + p - 1}.$$

Theorem 4.1 (Zubrilina [37, Theorem 1]). Let X, Y, P be parameters going infinite with $X, Y > 0$ and P prime; assume further that $Y = (1 + o(1))X^{1-\delta_2}$ and $P \ll X^{1+\delta_1}$ for some δ_1, δ_2 with $0 < \delta_1 < \frac{1}{11}, 2\delta_1 < \delta_2 < \frac{1}{13}(4 - 18\delta_1)$. Let $y = P/X$. Then

$$\mathbb{E}_{\substack{N \in [X, X+Y] \\ N \text{ squarefree} \\ f \in H^{\mathrm{new}}(N,k)}} [\sqrt{P}\lambda_f(P)\epsilon(f)] = \frac{\sum^{\square}_{N \in [X,X+Y]} \sum_{f \in H^{\mathrm{new}}(N,k)} \sqrt{P}\lambda_f(P)\varepsilon(f)}{\sum^{\square}_{N \in [X,X+Y]} \sum_{f \in H^{\mathrm{new}}(N,k)} 1} = \mathcal{M}_k(y) + O_\varepsilon\left(X^{-\delta' + \varepsilon} + \frac{1}{P}\right) \tag{15}$$

where

$$\mathcal{M}_k(y) = \frac{\alpha(-1)^{k/2-1}}{k-1} \sum_{1 \leq r \leq 2\sqrt{y}} \nu(r)\sqrt{4y - r^2}\, U_{k-2}\left(\frac{r}{2\sqrt{y}}\right) + \frac{\beta}{k-1}\sqrt{y} - \gamma\delta_{k=2}\, y. \tag{16}$$

Here $\delta' > 0$ is a constant explicitly expressible through δ_1, δ_2. The notation $\sum^{\square}$ means the sum is over square-free integers.

Figure 5 shows the graphs of $\mathcal{M}_k(y)$ for $k = 2, 8, 24$ (note that $U_0(x) = 1$ and $\mathcal{M}_2(y)$ is a linear combination of the functions of the form $\sqrt{4y - r^2}$). They are continuous but not differentiable at $y = \frac{r^2}{4}$ for $r \in \mathbb{Z}_{>0}$, which are the points where the summation index r changes.

4.2 Skoruppa–Zagier trace formula

To prove Theorem 4.1, one needs to understand how to estimate the numerator on the LHS. Recall that $a_f(P)$ is the P-th Fourier coefficient of f, which is also the eigenvalue of the Hecke operator T_P acting on f. Also, $(-1)^{k/2}\varepsilon(f)$ is equal to the eigenvalue of the Atkin–Lehner involution $W_N = T_N$ acting on f. Thus, the sum appearing in the numerator on the LHS of (15) can be interpreted as the trace of the operator $(-1)^{k/2}T_P \circ W_N$ acting on the space of cusp forms of weight k and level N (multiplied by $P^{1-k/2}$). Eichler [8] studied such sums of traces and proved that they can be expressed in terms of (Hurwitz) class numbers, which was generalized by Selberg [26]. To account for the root number $\varepsilon(f)$, i.e., the eigenvalue of W_N, we use the following version of the Eichler–Selberg trace formula by Skoruppa and Zagier [31].

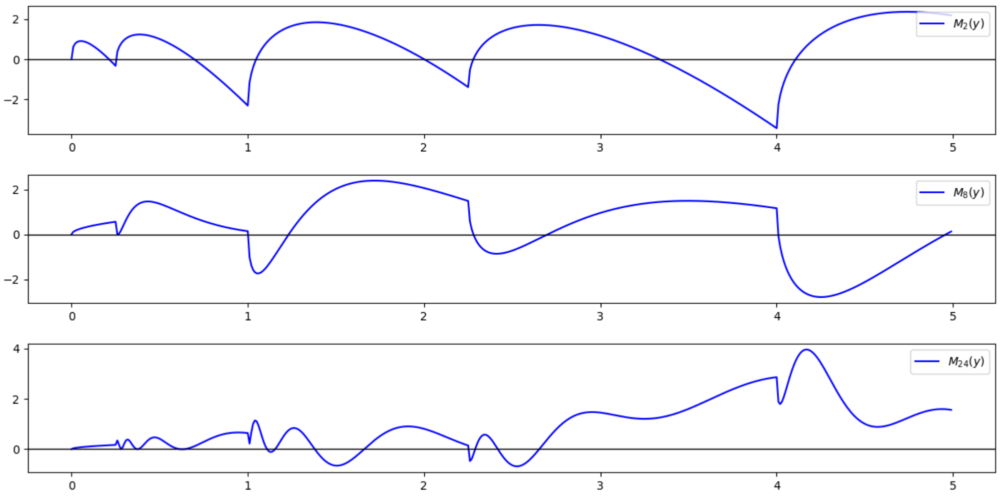


Figure 5: Murmuration density $\mathcal{M}_k(y)$ of modular forms for $k = 2, 8, 24$.

Theorem 4.2 (Skoruppa–Zagier [31])**.** For square-free N and prime $P \nmid N$,

$$\sum_{f \in H^{\text{new}}(N,k)} \sqrt{P}\lambda_f(P)\varepsilon(f) = \frac{H_1(-4PN)}{2} + (-1)^{k/2-1}U_{k-2}\left(\frac{r\sqrt{N}}{2\sqrt{P}}\right) \sum_{0<r\leq 2\sqrt{P/N}} H_1(r^2N^2 - 4PN) - \delta_{k=2}(P+1)$$

Here $H_1(-d)$ $(d > 0)$ is the Hurwitz class number, the number of equivalence classes of positive definite binary quadratic forms of discriminant $-d$ weighted by the number of automorphisms, i.e. with forms correspond to $x^2 + y^2$ or $x^2 + xy + y^2$ counted with multiplicity $1/2$ and $1/3$ respectively.

If we denote the spaces of eigenforms with root number $+1$ and -1 by $H_+^{\text{new}}(N,k)$ and $H_-^{\text{new}}(N,k)$ respectively, then the above theorem can be interpreted as the difference of the traces of T_P acting on $H_+^{\text{new}}(N,k)$ and $H_-^{\text{new}}(N,k)$, while the Eichler–Selberg trace formula gives the sum of the two traces. Theorem 4.2 is proven by using the theory of Jacobi forms, which we will not discuss here. Hurwitz class number can be expressed as a sum of usual class numbers as

$$H_1(-d) = \sum_{f^2|d} h(-d/f^2) + O(1)$$

where the "error term" $O(1)$ disappears if $d \neq 3 \cdot \square$ or $4 \cdot \square$. Using this, we can rewrite the Skoruppa–Zagier trace formula as

$$\sum_{f \in H^{\text{new}}(k,N)} \sqrt{P}\lambda_f(P)\varepsilon(f)$$
$$= \frac{h(-4PN)}{2} + \frac{h(-PN)}{2} - \delta_{k=2}P + O(1) + (-1)^{k/2-1}U_{k-2}\left(\frac{r\sqrt{N}}{2\sqrt{P}}\right) \sum_{1\leq r\leq 2\sqrt{P/N}} \sum_{d^2|r^2N-4P} h\left(\frac{N(r^2N-4P)}{d^2}\right)$$

From this, our new goal is to estimate the average of class numbers over short intervals, i.e. when $N \in [X, X+Y]$ with $Y = o(X)$. The main idea is to use class number formula to write class numbers as special L-values at $s = 1$, e.g.

$$h(-d) = \frac{\sqrt{d}}{\pi}L(1, \chi_d)$$

when $d > 4$ and $-d \not\equiv 2, 3 \pmod 4$, and $\chi_d = \left(\frac{d}{\cdot}\right)$ is the Kronecker symbol. Then the sum (average) of the corresponding L-values can be estimated via truncation and Polya–Vinogradov inequality (13). For example, we have an estimate

$$L(1, \chi_d) = \sum_{n \geq 1} \frac{\chi_d(n)}{n} = \sum_{1 \leq n \leq T} \frac{\chi_d(n)}{n} + O\left(\frac{\sqrt{d} \log d}{T}\right).$$

With some hard analysis, one gets the following estimations on the sum of $h(-PN)$ and $h(-4PN)$.

Proposition 4.3 (Zubrilina [37, Proposition 3.1])**.** Let P be an odd prime and let $[X, X + Y]$ be an interval with $Y = o(X)$. Then as $X \to \infty$,

$$\frac{\zeta(2)\pi}{XY} \sum_{\substack{N \in [X, X+Y] \\ P \nmid N}}^{\square} \left(\frac{h(-PN)}{2} + \frac{h(-4PN)}{2}\right) = A\sqrt{y} + O_\varepsilon\left(\frac{1}{P^{3/2}X^{1/2}} + \frac{P^{7/12}}{Y^{5/6}X^{5/12}} + \frac{YP^{1/2}}{X^{3/2}}\right)(XYP)^\varepsilon$$

where

$$A = \prod_p \left(1 + \frac{p}{(p+1)^2(p-1)}\right).$$

The summation of $H_1(r^2N^2 - 4PN)$ terms can be bounded in a similar way, although the computation is much more complicated.

Proposition 4.4 (Zubrilina [37, Proposition 3.2])**.** Let P be an odd prime, $r \in \mathbb{N}$, and $X > Y > 0$ be such that $r^2(X + Y) < 4P$ for each $r > 2\sqrt{P/X}$. Let $y = P/X$. Then

$$\frac{\zeta(2)\pi}{XY} \sum_{r \leq 2\sqrt{P/X}} \sum_{\substack{N \in [X, X+Y] \\ P \nmid N}}^{\square} H_1\left(r^2N^2 - 4PN\right)$$
$$= \sum_{r \leq 2\sqrt{P/X}} B\nu(r)\sqrt{4y - r^2} + O\left(\frac{P^{11/10}}{Y^{2/5}X^{9/10}} + \frac{YP}{X^2} + \frac{PY^{1/2}}{X^{3/2}} + \frac{P}{X^{1/2}Y^{13/18}} + \frac{P}{XY^{1/9}}\right)(XYP)^\varepsilon$$

where

$$B = \prod_p \frac{p^4 - 2p^2 - p + 1}{(p^2 - 1)^2}.$$

4.3 Geometric intervals

Theorem 4.1 considers the average over "short intervals" $[X, X + Y]$ with $Y = o(X)$. By integrating it in a suitable sense, one can also get the average over "geometric intervals" $[X, cX]$ for some constant $c > 1$.

Theorem 4.5 ([37, Theorem 2])**.** Let $P \ll X^{6/5}$, $c > 1$ be a constant, and $y = P/X$. As $X \to \infty$,

$$\mathbb{E}_{\substack{N \in [X, cX] \\ N \text{ squarefree} \\ f \in H^{\text{new}}(N,k)}} [\sqrt{P}\lambda_f(P)\epsilon(f)] = \frac{2}{c^2 - 1} \int_1^c u\mathcal{M}_k\left(\frac{y}{u}\right) \mathrm{d}u + o_y(1)$$

where $\mathcal{M}_k(y)$ is as in Theorem 4.1. In particular, for $k = c = 2$, the dyadic average

$$\frac{\sum_{N \in [X, 2X]}^{\square} \sum_{f \in H^{\text{new}}(N,2)} a_f(P)\epsilon(f)}{\sum_{N \in [X, 2X]}^{\square} \sum_{f \in H^{\text{new}}(N,2)} 1}$$

converges to

$$\begin{cases} a\sqrt{y} - by & 0 \le y \le \frac{1}{4} \\ a\sqrt{y} - by + c\pi y^2 - c(1-2y)\sqrt{y - \frac{1}{4}} - 2cy^2 \arcsin\left(\frac{1}{2y} - 1\right) & \frac{1}{4} \le y \le \frac{1}{2} \\ a\sqrt{y} - by + 2cy^2\left(\arcsin\left(\frac{1}{y} - 1\right) - \arcsin\left(\frac{1}{2y} - 1\right)\right) & \\ \quad -c(1-2y)\sqrt{y - \frac{1}{4}} + 2c(1-y)\sqrt{2y-1} & \frac{1}{2} \le y \le 1 \end{cases}$$

where $a = \frac{4}{9}(2^{3/2} - 1)\beta, b = \frac{2}{3}\gamma, c = \frac{2}{3}\alpha$. for explicit constants a, b, c.

Proof. The main idea of the proof is to divide the interval $[X, cX]$ into short intervals $[X_g, X_{g+1}]$ for $X_g = X + (g-1)Y$ where $Y \sim X^{1-\delta_2}$. Then use Theorem 4.1 to approximate the sum over short intervals as an integral of $u\mathcal{M}_k(y/u)$. The case of $k = c = 2$ can be done by elementary computations. □

The murmuration density function $\mathcal{M}_k(y)$ has many interesting properties. Especially, their properties can be related to Katz and Sarnak's 1-level density conjecture [16]; see Section 7 for more details.

4.4 Doesn't Zubrilina's result prove murmuration for elliptic curves, because of the modularity?

No! The reason is that elliptic curves over $\mathbb{Q}$ correspond to modular forms of weight 2 *with coefficient field (Hecke field)* $\mathbb{Q}$ (recall that $a_p(E) = p + 1 - \#E(\mathbb{F}_p)$ are integers). The family (see also Section 7) that Zubrilina considered is much larger than the family of elliptic curves in [13], and it seems hard to isolate such a family from the whole family of Hecke eigenforms (of weight 2). It is conjectured that the *conductor dimension* (See 7 for the definition) of elliptic curves is $\frac{5}{6}$ [28], while that of the weight 2 modular forms is 2.

4.5 Without root number

In [21, 22], Martin considered murmurations without the root number weight but with $\sqrt{N}$ scaling, i.e. the average of the form

$$\mathbb{E}_{\substack{N \in [X, cX] \\ N \text{ squarefree} \\ f \in H^{\text{new}}(N,k)}} [\sqrt{NP}\lambda_f(P)] = \frac{\sum^{\square}_{N \in [X,cX]} \sum_{f \in H^{\text{new}}(N,k)} \sqrt{NP}\lambda_f(P)}{\sum^{\square}_{N \in [X,cX]} \sum_{f \in H^{\text{new}}(N,k)} 1} \tag{17}$$

or more generally,

$$\mathbb{E}_{\substack{N \in [X, cX] \\ f \in H^{\text{new}}(N,k)}} \left[w_Q(f) \sqrt{\frac{NP}{Q}} \lambda_f(P) \right] \tag{18}$$

for a sequence of divisors $Q \mid N$ for each N appearing in the average, where $w_Q(f)$ is the eigenvalue of the Atkin–Lehner involution W_Q acting on f. $Q = N$ corresponds to the Zubrilina's case (Theorem 4.5), while $Q = 1$ reduces to (17). He conjectured that we do have murmurations in such general cases for "arithmetically compatible sequences of (N, Q)" [21, Conjecture 1.8], and computed the murmuration density for small $y = P/X$ (See also Figure 6). The analogous trace formula for $T_P \circ W_Q$ is given in [21, Proposition 5.5], although the analysis would be much harder since the number of class number terms grows with P (it is bounded in (17)).

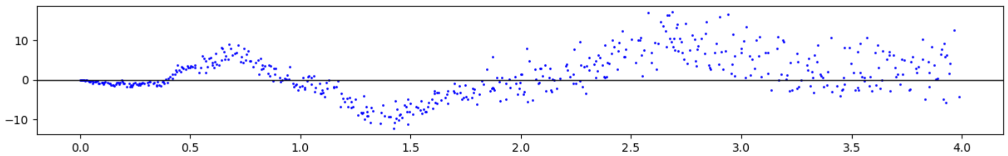


Figure 6: Murmuration of modular forms without root number weight for $k = 2$, $N \in [1000, 2000]$ and $P < 4000$.

5 Murmuration of Elliptic Curves, Revisited

Recently, Will Sawin and Andrew Sutherland announced a murmuration theorem for elliptic curves [25], which is slightly different from the formulation in [13]. Especially, they proved a version of the murmuration theorem *ordered by height*:

Theorem 5.1 (Sawin–Sutherland [25])**.** Let

$$\mathcal{E}(X) := \{y^2 = x^3 + ax + b : a, b \in \mathbb{Z}, p^4 \mid a \Rightarrow p^6 \nmid b, \max\{4|a|^3, 27b^2\} \leq X\}$$

be the set of isogenous classes of elliptic curves over $\mathbb{Q}$ ordered by naive height. For any smooth function $W : \mathbb{R}_{>0} \to \mathbb{R}$ with compact support, the limit

$$\lim_{P\to\infty} \lim_{X\to\infty} \mathbb{E}_{E\in\mathcal{E}(X)} \left[\frac{\prod_{p\leq P}(1-p^{-1})^{-1}}{N(E)} \sum_{\substack{n\geq 1\\ p\nmid n\ \forall p\leq P}} W\left(\frac{n}{N(E)}\right) a_n(E)\epsilon(E) \right] \tag{19}$$

exists and is equal to

$$\int_0^\infty W(u)\sqrt{u}\left(2\pi \sum_{q\geq 1}^{\square} \sum_{m\geq 1} \frac{\mu(\gcd(m,q))}{qm\phi\left(\frac{q}{\gcd(m,q)}\right)} J_1\left(\frac{4\pi\sqrt{u}m}{q}\right) \prod_{p\mid q} \hat{\ell}_{p,2v_p(m)} \prod_{p\mid m, p\nmid q} \ell_{p,2v_p(m)}\right) \mathrm{d}u \tag{20}$$

where $\ell_{p,v}$ and $\hat{\ell}_{p,v}$ are certain local factors that can be written in terms of traces of the Hecke operator T_p (see [25, Lemma 3, 4]).

They also conjectured the following:

Conjecture 5.2 ([25, Conjecture 1])**.** For any $0 < C_1 < C_2$, we have

$$\lim_{X\to\infty} \mathbb{E}_{E\in\mathcal{E}(X)} \left[\frac{\log\left(N(E)\frac{C_1+C_2}{2}\right)}{N(E)} \sum_{p\in(C_1N(E), C_2N(E))} \epsilon(E)a_p(E) \right]$$
$$= \int_{C_1}^{C_2} 2\pi\sqrt{u} \sum_{q\geq 1}^{\square} \sum_{m\in\mathbb{N}} \frac{\mu(\gcd(m,q))}{qm\phi\left(\frac{q}{\gcd(m,q)}\right)} J_1\left(\frac{4\pi\sqrt{u}m}{q}\right) \prod_{p\mid q} \hat{\ell}_{p,2v_p(m)} \prod_{p\mid m, p\nmid q} \ell_{p,2v_p(m)} \mathrm{d}u$$

You should have a question at this point. We said that Sutherland observed no murmuration pattern in [33] when elliptic curves are ordered by height, but Theorem 5.1 seems to suggest that there is a murmuration pattern. In fact, the difference comes from *local averaging*, which I'm going to explain now.

The difference between the original murmuration observed in HLOP [13] and the one in Sawin–Sutherland is well-explained in [25, Section 1.1]. The original murmuration considered the averages of the form

$$\mathbb{E}_{\substack{N(E)\in[N_1,N_2]\\ \mathrm{rank}(E)=r}}[a_p(E)]$$

as a function in p for fixed r, N_1, N_2 (initially $[N_1, N_2] = [7500, 10000]$ in [13]). As mentioned earlier, subsequent works (especially [33]) found that we need to view the murmuration density as a function in p/N, not p. Also, it seems better to consider all elliptic curves with same root numbers at once, or weight a_p by root numbers. Hence the reformulated HLOP's murmuration would be

$$\mathbb{E}_{N(E)\in[X,2X]}[\epsilon(E)a_p(E)]$$

In [25], the authors mentioned that Bober suggested that one may need *local averaging* in p before we average over different elliptic curves, and consider the following double average:

$$\mathbb{E}_{N(E)\in[X,2X]}\left[\mathbb{E}_{\substack{p\in(C_1N(E),C_2N(E))\\ p \text{ prime}}}[\epsilon(E)a_p(E)]\right]$$

It seems analytically better to work with the approximated version

$$\mathbb{E}_{N(E)\in[X,2X]}\left[\frac{\log\left(N(E)\frac{C_1+C_2}{2}\right)}{N(E)}\sum_{p\in(C_1N(E),C_2N(E))}\epsilon(E)a_p(E)\right]$$

and replacing the outer conductor ordering by naive height ordering gives Conjecture 5.2, where Theorem 5.1 is a variant of the conjecture where primes are replaced by all $n \geq 1$ and the weight function is smooth with compact support. Heuristics like Crámer's random model suggests that these changes do not affect the density function.

The main idea of the proof of Theorem 5.1 is the following Voronoi summation formula, where the summation over n instead of primes is built-in inside the formula.

Theorem 5.3 ([25, Lemma 11])**.** Let $E_{/\mathbb{Q}}$ be an elliptic curves, q be a positive integer, a a positive integer coprime to q, and $W : (0, \infty) \to \mathbb{R}$ a smooth function with compact support. Then

$$\epsilon(E)\sum_{n\geq 1}\frac{a_n(E)}{\sqrt{n}}W\left(\frac{n}{N(E)}\right)e\left(\frac{an}{q}\right) = \frac{\sqrt{N(E)}}{q}\sum_{n\geq 1}\frac{a_n(E)}{\sqrt{n}}e\left(\frac{\overline{aN(E)}n}{q}\right)\int_0^\infty 2\pi W(u)J_1\left(\frac{4\pi\sqrt{u}n}{q}\right)\mathrm{d}u \tag{21}$$

where $e(x) = e^{2\pi ix}$ and $\overline{aN(E)}$ is the multiplicative inverse of $aN(E)$ modulo q.

You can find more about the work from Sutherland's lecture [34] at Tate conference (*The legacy of John Tate, and beyond* at Harvard university). He considered it as *a* murmuration theorem, and might not be *the* murmuration theorem since the density formula is too complicated.

6 Other known cases

At last, we mention several other families where murmuration densities have been computed.

6.1 Flying Hecke characters of imaginary quadratic fields

Wang [36] computed the murmuration density for (possibly trivial) Hecke characters of imaginary quadratic fields. Let $\mathcal{F}$ be the family of nontrivial Hecke characters of $\mathbb{Q}(\sqrt{-D})$ for square-free $D > 3$, $D \equiv 3 \pmod 4$. For $\psi \in \mathcal{F}$, let $a_\psi(p) = \sum_{\mathrm{N}\mathfrak{p}=p} \psi(\mathfrak{p})$ be the trace of Frobenius at p. The completed L-function $\Lambda(s, \psi)$ satisfies the functional equation $\Lambda(1-s, \psi) = \Lambda(s, \psi)$, so the root number is always $+1$.

Theorem 6.1 (Wang [36])**.** Then the average of normalized trace $\sqrt{p}a_\psi(p)$ over $\psi \in \mathcal{F}$ with $N_\psi = |D_\psi| \in [X, X+Y]$ is

$$\mathbb{E}_{\substack{\psi\in\mathcal{F}\\ N_\psi\in[X,X+Y]}}[\sqrt{p}a_\psi(p)] = \frac{\sum_{\substack{\psi\in\mathcal{F}\\ N_\psi\in[X,X+Y]}} \sqrt{p}a_\psi(p)}{\sum_{\substack{\psi\in\mathcal{F}\\ N_\psi\in[X,X+Y]}} 1} = c(p) \sum_{1\le m<2\sqrt{y}} \delta_m(p) M_m(y) + M_-(y) + \text{error} \tag{22}$$

where

$$M_m(y) = \frac{11\zeta(2)}{4A}\sqrt{\frac{y}{4y-m^2}}\vartheta(m), \quad \vartheta(m) = 2^{\omega(m)+\min\{v_2(m),2\}} \prod_{\substack{q|y\\ q \text{ odd}}} \left(1 + \frac{2q^2+q-1}{q^4-3q^2-2q+2}\right)$$

$$M_-(y) = -\frac{11\pi}{12A}\sqrt{y}$$

$$c(p) = \frac{p+1}{3p} \prod_{\ell>2, \left(\frac{p}{\ell}\right)=1} \left(1 - 2\ell^{-2} - \frac{2\ell^{-3}}{1-\ell^{-2}}\right)$$

$$\delta_m(p) = \begin{cases} \mathbb{1}_{\left(\frac{p}{q}\right)=1} & m = q^k,\ q \text{ is odd prime} \\ \mathbb{1}_{p\equiv 3 \pmod 4} & m = 2 \\ \mathbb{1}_{p\equiv 5 \pmod 4} & m = 4 \\ \mathbb{1}_{p\equiv 1 \pmod 8} & m = 2^\nu,\ \nu \ge 3 \end{cases}$$

See [36, Theorem 1] for details. These characters give rise to modular forms of weight 1 via theta series, and it is conjectured that almost all weight 1 modular forms arise this way [10]. Note that the main term of (22) depends on the arithmetic of p. However, the dependence on p is explicit and $c(p)\delta_m(p)$ is almost periodic in m, a phenomenon that does not appear in other families. In particular, he proved that taking a local average over short intervals of p removes the dependence on p. Especially, taking a local average in an interval of length H so that the primes are equidistributed in $[X, X+H]$, in the sense that

$$\sum_{\substack{X\le n\le X+H\\ n\equiv a \pmod q}} \Lambda(n) = \frac{H}{\varphi(q)}(1+o(1))$$

where Λ is the von Mangoldt function (for example, $H = X^{\frac{1}{2}+\epsilon}$ is enough under RH), then we get the following theorem.

Theorem 6.2 (Wang [36, Theorem 2])**.** Let $P \sim X^{1+\delta_p}$ be a prime with fixed $\delta_p \ge 0$. We have

$$\mathbb{E}_{p\in[P,P+H]}\left[\mathbb{E}_{\substack{\psi\in\mathcal{F}\\ N_\psi\in[X,X+Y]}}\left[\sqrt{p}a_\psi(p)\right]\right] = \bar{c} \sum_{1\le m<2\sqrt{y}} \overline{M}_m(y) + M_-(y) + o(1) \tag{23}$$

where

$$\bar{c} = \frac{1}{3}\prod_{\ell>2}\left(1-\ell^{-2}-\frac{\ell^{-3}}{1-\ell^{-2}}\right),$$

$$\overline{M}_m(y) = \frac{11\zeta(2)}{4A}\sqrt{\frac{y}{4y-m^2}}\kappa(m), \quad \kappa(m) = 2^{\mathbb{1}_{2|m}}\prod_{\substack{q|m \\ q \text{ odd}}}\left(1+\frac{q^2}{q^4-2q^2-q+1}\right)$$

and $M_-(y)$ is the same as in Theorem 6.1.

The main ingredients of the proof are orthogonality of characters and summation of class numbers in short intervals with the class number formula, similar to Zubrilina's approach.

6.2 Flying modular forms (in weight direction)

Recall that Zubrilina computed the murmuration density for a *fixed weight k and varying level N*. In [3], Bober, Booker, M. Lee, and Lowry-Duda considered the opposite case, where they fix the level $N = 1$ and vary the weight k. In this case, they considered the family of Hecke newforms whose *analytic conductor*

$$\mathcal{N}(k) := \left(\frac{\exp\psi(k/2)}{2\pi}\right)^2 \approx \left(\frac{k-1}{4\pi}\right)^2 + O(1)$$

are in certain range, where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function.

Theorem 6.3 (Bober–Booker–Lee–Lowry-Duda [3, Theorem 1.1])**.** Fix $\epsilon \in (0, \frac{1}{12})$, $\delta \in \{0, 1\}$, and a compact interval $E \subset \mathbb{R}_{>0}$ with $|E| > 0$. Let $K, H > 0$ with $K^{\frac{5}{6}+\epsilon} < H < K^{1-\epsilon}$, and let $N = \mathcal{N}(K)$. Then as $K \to \infty$, we have

$$\frac{\sum_{\substack{p \text{ prime} \\ p/N\in E}}\log p\sum_{\substack{k\equiv 2\delta \,(\mathrm{mod}\, 4) \\ |k-K|\le H}}\sum_{f\in H_k(1)}\lambda_f(p)}{\sum_{\substack{p \text{ prime} \\ p/N\in E}}\log p\sum_{\substack{k\equiv 2\delta \,(\mathrm{mod}\, 4) \\ |k-K|\le H}}\sum_{f\in H_k(1)}1} = \frac{(-1)^\delta}{\sqrt{N}}\left(\frac{\nu(E)}{|E|} + o_{E,\epsilon}(1)\right) \tag{24}$$

where

$$\nu(E) = \frac{1}{\zeta(2)}\sideset{}{^*}\sum_{(a/q)^{-2}\in E}\frac{\mu(q)^2}{\varphi(q)^2\sigma(q)}\left(\frac{a}{q}\right)^{-3} = \frac{1}{2}\sum_{t\in\mathbb{Z}}\prod_{p\nmid t}\frac{p^2-p-1}{p^2-p}\int_E\cos\left(\frac{2\pi t}{\sqrt{y}}\right)\mathrm{d}y \tag{25}$$

where the summation $\sum^*$ indicates that the terms occuring at the endpoints of E are halved.

The main tool for the proof is the (original) Eichler–Selberg trace formula, which does not include Atkin–Lehner operators (e.g. [5, Theorem 2.1]). Then, they apply the class number formula to replace class numbers with the special values of Dirichlet L-functions at $s = 1$, which can be estimated under GRH.

6.3 Flying Maass forms

Booker, Lee, Lowry-Duda, Seymour-Howell, and Zubrilina computed murmuration densities for weight 0 and level 1 Maass forms [4]. They considered a family of Maass forms where the spectral parameter (R with $\lambda = \frac{1}{4} + R^2$) goes to ∞, which is equivalent to the *analytic conductor*

$$\mathcal{N}(R) := \frac{\exp\left(\psi\left(\frac{1/2+a+iR}{2}\right)\right) + \exp\left(\psi\left(\frac{1/2+a-iR}{2}\right)\right)}{\pi^2} \approx \frac{R^2}{4\pi^2} + O(1)$$

going to ∞. Here $a = 0$ (resp. $a = 1$) if the Maass form is even (resp. odd).

Theorem 6.4 (Booker–Lee–Lowry-Duda–Seymour-Howell–Zubrilina [4, Theorem 1.1])**.** Let $E \subset \mathbb{R}_{>0}$ be a fixed compact interval with $|E| > 0$. Let $R, H > 0$ with $R^{\frac{5}{6}+\delta} < H < R^{1-\delta}$ for some $\delta > 0$ and $N = \mathcal{N}(R)$. Assuming GRH for L-functions of Dirichlet characters and Maass forms, as $R \to \infty$ we have

$$\frac{\sum_{\substack{p \text{ prime} \\ p/N \in E}} \log p \sum_{|r(f)-R|\le H} \epsilon(f) a_f(p)}{\sum_{\substack{p \text{ prime} \\ p/N \in E}} \log p \sum_{|r(f)-R|\le H} 1} \to \frac{(-1)^\delta}{\sqrt{N}} \left(\frac{\nu(E)}{|E|} + o_{E,\epsilon}(1) \right) \tag{26}$$

where $\nu(E)$ is the same as (25) of Theorem 6.3.

Proof uses an explicit Selberg trace formula due to Str"ombergsson in his unpublished work [32], which requires an analytic test function and cannot be compactly supported, where GRH is needed to control the cutoff error term. The remainder of the proof is similar to the weight aspect case of the modular forms [3].

7 General formulation of Murmuration

In his letter to Sutherland and Zubrilina, Sarnak proposed a general framework of murmuration [23] for *families* of L-functions. We explain his definition and its connection with the Katz–Sarnak philosophy [16, 17]. Also, we revisit previous murmuration results [37, 18, 25, 36] in this framework. See also Lowry-Duda's survey note [19].

7.1 Murmuration for families

Let $\mathcal{F}$ be a family of L-functions in a suitable sense (e.g. See [24]). For a smooth nonnegative function $\Phi : (0, \infty) \to \mathbb{R}$ with compact support and $f : \mathcal{F} \to \mathbb{C}$, consider the Φ-weighted average of f:

$$\mathbb{E}_{\pi \in \mathcal{F}}[f; \Phi, N] := \frac{\sum_{\pi \in \mathcal{F}} \Phi\left(\frac{N_\pi}{N}\right) f(\pi)}{\sum_{\pi \in \mathcal{F}} \Phi\left(\frac{N_\pi}{N}\right)} = \frac{A_{\mathcal{F}}(f; \Phi, N)}{A_{\mathcal{F}}(1; \Phi, N)} \tag{27}$$

where

$$A_{\mathcal{F}}(f; \Phi, N) := \sum_{\pi \in \mathcal{F}} \Phi\left(\frac{N_\pi}{N}\right) f(\pi). \tag{28}$$

Here N_π is the "conductor" of π (e.g. conductor of an elliptic curve or analytic conductor of an automorphic form). When we order the family by the conductor, we say that $\mathcal{F}$ has *conductor dimension* δ if

$$\#\{\pi \in \mathcal{F} : N_\pi \le N\} \sim \alpha N^\delta \tag{29}$$

as $N \to \infty$ for some $\alpha > 0$ and $\delta = \delta(\mathcal{F}) > 0$. For such family, we have

$$A_{\mathcal{F}}(1; \Phi, N) \sim \alpha \delta N^\delta \int_0^\infty \Phi(x) x^\delta \frac{\mathrm{d}x}{x}.$$

Most of the known murmuration results consider the function

$$f(\pi) = a_\pi(p) := \sqrt{p} \lambda_\pi(p) \tag{30}$$

for a given prime p, where $\lambda_\pi(p)$ is the normalized trace of Frobenius at p so that the Ramanujan–Petersson conjecture says $|\lambda_\pi(p)| \le n$ for GL_n automorphic forms π. Furthermore, if $\mathcal{F}$ is self-dual, then $a_\pi(p)$ is real and the global root number w_π is either 1 or -1. Then we can separate by root number and consider the averages

$$\mathbb{E}_{\pi\in\mathcal{F}^w}[a_\pi(p); \Phi, N] \tag{31}$$

for $w \in \{\pm 1\}$ and $\mathcal{F}^w = \{\pi \in \mathcal{F} : w_\pi = w\}$.

Definition 7.1. A continuous function $M_\Phi : (0, \infty) \to \mathbb{R}$ is a *murmuration function for $\mathcal{F}$ with weight Φ* if there is $0 \le \gamma < 1$ such that for $P \sim N$

$$\mathbb{E}_{\pi\in\mathcal{F}}[a_\pi(p); \Phi, N] = M_\Phi\left(\frac{p}{N}\right) + R(p, N) \tag{32}$$

where the local oscillating term $R(p, N)$ satisfies[2]

$$\mathbb{E}_{P\le p\le P+H}[R(p, N)] = \frac{\sum_{P\le p\le P+H} R(p, N)}{\sum_{P\le p\le P+H} 1} = o(1) \tag{33}$$

for $H = o(N)$.

Equation (33) is the local averaging over p of length at least N^γ and less than N. The smaller γ we can take, the more visible M_Φ is. In particular, $\gamma = 0$ means that no local averaging is needed. He conjectured that if

$$\delta + \gamma > 1 \tag{34}$$

then the local oscillating term $R(p, N)$ will vanish as $N \to \infty$. In particular, we may not need local averaging if $\delta > 1$.

The function M_Φ is supposed to have a form of

$$M_\Phi(y) = \frac{\int_0^\infty \Phi(u) M\left(\frac{y}{u}\right) u^\delta \frac{\mathrm{d}u}{u}}{\int_0^\infty \Phi(u) u^\delta \frac{\mathrm{d}u}{u}}. \tag{35}$$

for some universal $M : (0, \infty) \to \mathbb{R}$. If such a function (more generally, distribution) exists, we call it as *Zubrilina murmuration density for $\mathcal{F}$*, denoted as $Z_\mathcal{F}$. It might be a distribution on $C_c^\infty((0, \infty))$ rather than a function.

7.2 Katz–Sarnak philosophy

Katz and Sarnak [16, 17] studied statistics of zeros of L-functions via random matrix models. In particular, they considered *one-level density* of low-lying zeros: for an even function ϕ with rapid decay as $|x| \to \infty$, the one-level density of a family $\mathcal{F}$ is

$$\mathrm{OLD}(\mathcal{F}; \phi) = \lim_{N\to\infty} \mathbb{E}_{\pi\in\mathcal{F}(N)}\left[\sum_{\gamma_\pi} \phi\left(\frac{\gamma_\pi \log N}{2\pi}\right)\right] \tag{36}$$

[2]In [23], he considered local average over primes $P - H \le p \le P + H$ instead of $P \le p \le P + H$, but most of the latter works seems to follow one-sided averaging.

where $\mathcal{F}(N) := \{\pi \in \mathcal{F} : N_\pi = N\}$ and γ_π runs through the ordinates of nontrivial zeros of $L(s,\pi)$ on the critical line, i.e. $L\left(\frac{1}{2} + i\gamma_\pi, \pi\right) = 0$. The factor $\frac{\log N}{2\pi}$ guarantees that the nontrivial zeros have unit spacing on average. Katz–Sarnak philosophy claims that there is a measure $W_\mathcal{F}$ coming from matrices related to the "type" of $\mathcal{F}$ such that

$$\mathrm{OLD}(\mathcal{F};\phi) = \int_\mathbb{R} \widehat{\phi}(x)\widehat{W_\mathcal{F}}(x)\mathrm{d}x \tag{37}$$

for any nice test function ϕ. We say that $\mathcal{F}$ has orthogonal/symplectic/unitary symmetry type if $W_\mathcal{F}$ comes from one of the orthogonal/symplectic/unitary groups. More precisely, let $G(N)$ be one of the "ensembles" in [16, Table 1], e.g. $G(N) = \mathrm{U}(N)$, the compact group of $N \times N$ unitary matrices. Let $\mathrm{d}A$ be a Haar measure on $G(N)$, and let $e^{i\theta_1(A)}, \ldots, e^{i\theta_N(A)}$ be the eigenvalues of $A \in G(N)$, where $0 \le \theta_1(A) \le \cdots \le \theta_N(A) < 2\pi$. Let

$$\Delta(A)[a,b] := \#\{\theta(A) : e^{i\theta(A)} \text{ is an eigenvalue of } A \text{ and } \frac{\theta(A)N}{2\pi} \in [a,b]\}.$$

Then we have a density function $W_G : \mathbb{R} \to \mathbb{R}_{\ge 0}$ such that for any interval $[a,b] \subset \mathbb{R}$,

$$\lim_{N\to\infty} \int_{G(N)} \Delta(A)[a,b]\mathrm{d}A = \int_a^b W_G(x)\mathrm{d}x.$$

For example, W_G for unitary/orthogonal/symplectic groups are given by

$$W_\mathrm{U}(x) = 1, \tag{38}$$

$$W_{\mathrm{SO}(+)(x)} = 1 + \frac{\sin(2\pi x)}{2\pi x}, \tag{39}$$

$$W_{\mathrm{SO}(-)(x)} = 1 - \frac{\sin(2\pi x)}{2\pi x} + \delta_0(x), \tag{40}$$

$$W_\mathrm{Sp}(x) = 1 - \frac{\sin(2\pi x)}{2\pi x}. \tag{41}$$

One such example is the following theorem on the family of Hecke eigenforms by Iwaniec, Luo, and Sarnak [15], showing that the family has orthogonal symmetry type.

Theorem 7.2 (Iwaniec–Luo–Sarnak [15]). Assume GRH. Let ϕ be an even Schwartz function with $\mathrm{supp}(\widehat{\phi}) \subset (-2,2)$. Let $H_k^\pm$ be a set of Hecke eigenforms of weight k and root number $\epsilon = \pm 1$. Then

$$\mathrm{OLD}(H_k^\pm;\phi) = \int_\mathbb{R} \widehat{\phi}(x)\widehat{W_{\mathrm{SO}(\pm)}}(x)\mathrm{d}x \tag{42}$$

There is an explicit formula relating the summation over zeros of L-functions and over primes, which is given by [15, Section 4]

$$\sum_{\gamma_\pi} \phi\left(\frac{\gamma_\pi \log N}{2\pi}\right) = C - 2\sum_p \sum_{\nu\ge 1}\left(\sum_j \alpha_j(p)^\nu\right)\widehat{\phi}\left(\frac{\nu\log p}{\log N}\right)\frac{\log p}{p^{\nu/2}\log N}$$

and since $\widehat{\phi}$ is compactly supported, the main contribution comes from $\nu = 1$ summand, so we are mostly interested in

$$\sum_p \frac{\lambda_\pi(p)}{p^{1/2}}\widehat{\phi}\left(\frac{\log p}{\log N}\right)\frac{\log p}{\log N} \tag{43}$$

The Fourier transforms of (38) are

$$\widehat{W_{\mathrm{SO}(+)}}(y) = \delta_0(y) + \frac{2 - \mathbb{1}_{[-1,1]}(y)}{2}, \quad \widehat{W_{\mathrm{SO}(-)}}(y) = \delta_0(y) + \frac{\mathbb{1}_{[-1,1]}(y)}{2} \tag{44}$$

and there are obvious discontinuities at $y = \pm 1$. Hence, at least roughly, this suggests that the behavior of the average of $\lambda_\pi(p)/p^{1/2}$ with $p \sim N^a$ changes at $a = 1$. Katz–Sarnak philosophy suggests that, when $\mathcal{F}$ is of symplectic type, then

$$\mathbb{E}_{p \sim N^a}\left[\mathbb{E}_{\pi \in \mathcal{F}}[a_\pi(p); \Phi, N]\right] \to \begin{cases} 0 & a < 1 \\ -\frac{1}{2} & a > 1 \end{cases}$$

and when $\mathcal{F}$ is of orthogonal type,

$$\mathbb{E}_{p \sim N^a}\left[\mathbb{E}_{\pi \in \mathcal{F}^w}[a_\pi(p); \Phi, N]\right] \to \begin{cases} 0 & a < 1 \\ \frac{w}{2} & a > 1 \end{cases}$$

(see [23, eq. (7′)]). Each $a < 1$ and $a > 1$ corresponds to the limit of $M_\Phi(y)$ as $y \to 0$ and $y \to \infty$.

7.3 Revisiting the murmuration theorems

Let's see how the existing works on murmurations fit into the above framework.

7.3.1 Dirichlet characters

In case of Dirichlet characters [18], the family of even/odd *complex* Dirichlet characters are not self-dual, so it does not fit into Sarnak's framework precisely. Still, we can observe and prove murmuration phenomena for such a family, where the density function is much simpler compared to other families. Also, for each prime P, there are $\phi(P) = P - 1$ primitive characters modulo P, so there are about $\frac{CN^2}{\log N}$ primitive Dirichlet characters of prime conductors $\le N$ for some constant $C > 0$, which means that the conductor dimension of the family is $2 - \epsilon$[3]. In particular, (34) suggests that we don't need any further local averaging for the family, which is indeed the case of Theorem 3.1. Note that normalization used in Theorem 3.1 is also different from the one Sarnak suggested.

For the quadratic characters, local averaging is included in Theorem 3.2 over X^γ-many primes with $\frac{3}{4} < \gamma < 1$. Since the number of square-free odd numbers up to X is $\Theta(X)$, the conductor dimension of the family $\mathcal{F} = \{\chi_{8d} : d \text{ odd and square-free}\}$ is $\delta = 1$ so we may only need local averaging over X^ϵ-many primes for any $\epsilon > 0$ from (34). From Theorem 3.2, the Zubrilina density for the family is

$$Z_{\mathcal{F}}(y) = \frac{\sqrt{2}}{2} \sum_{\substack{a \ge 1 \\ \gcd(a,2)=1}} \frac{\mu(a)}{a^2} \sum_{m \ge 1} (-1)^m \cos\left(\frac{\pi m^2}{a^2 y} - \frac{\pi}{4}\right). \tag{45}$$

Note that this is not an actual function but rather a distribution, since the inner sum in m diverges for the most of y's. It seems that the distribution is a certain combination of delta functions at rational points, although it is not mentioned in [18]. Figure 7 shows that it might be supported at $0, \frac{1}{5}, \frac{3}{5}, \frac{4}{5}, 1$, etc.

[3] The $\log N$ factor disappears if we consider all conductors that are not necessarily prime, and the conductor dimension becomes 2. See [18, Section 6] for the version including composite conductors.

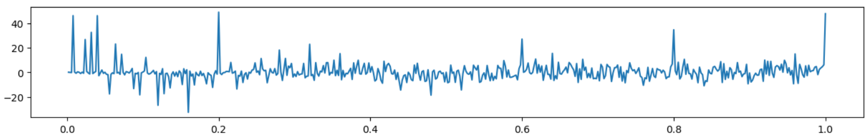


Figure 7: Plot of $Z_{\mathcal{F}}(y)$ for $y = \frac{i}{500}$ with $1 \le i \le 500$, truncated at $a < 200$ and $m \le 100$.

7.3.2 Modular forms

It is known that the dimension of the space of newforms of fixed weight k and level $\Gamma_0(N)$ is asymptotically $\frac{45(k-1)N}{\pi^2}$ [20, Theorem 8]. It implies that the number of newforms of fixed weight k and level $\Gamma_0(M)$ for square-free $M \le N$ is asymptotically $c_k N^2$ for a constant $c_k > 0$, so the conductor dimension of the family is $2 > 1$. From Sarnak's suggestion (34), we expect that we don't need any further local averaging, and this is indeed the case of Theorem 4.1 and 4.5.

Also, she studied several properties of her density function $\mathcal{M}_k$. In particular, for a compactly supported smooth function Φ on $(0, \infty)$, she proved that the function

$$\mathcal{M}_\Phi^k(y) := \frac{\int_0^\infty \mathcal{M}_k\left(\frac{y}{u}\right)\Phi(u)u^2\frac{\mathrm{d}u}{u}}{\int_0^\infty \Phi(u)u^2\frac{\mathrm{d}u}{u}} \tag{46}$$

is continuous in y and

$$\lim_{y\to 0^+}\mathcal{M}_k(y) = 0, \quad \lim_{y\to\infty}\mathcal{M}_k(y) = \frac{1}{2}.$$

This is compatible with Katz–Sarnak philosophy, since the family of modular forms has orthogonal symmetry type.

7.3.3 Elliptic curves

The conductor dimension of a family of elliptic curves ordered by *conductor* is conjecturally $\frac{5}{6}$ [28], so we may need local averaging over $X^{1/6+\epsilon}$-many primes from (34), although it is clear that there's an obvious murmuration pattern in the case (that's the first ever murmuration oberved in number theory!). In fact, it seems that the noise may still exist even if we increase the conductor range, as partially shown in Figure 2 (Sutherland computed murmuration up to the range of $[250000, 500000]$ [33], which is not reproduced in Figure 2 due to the computational limitation). Figure 8 shows the murmuration of non-CM elliptic curves with conductor in $[2^k, 2^{k+1})$ and primes $p < 2^k$ for $k = 12, \dots, 16$ weighted by root numbers, without (resp. with) local averaging of $X^\gamma = X^{\frac{1}{2}}$-many primes. Smaller γ gives less visible (purple) curves with larger noise.

In the case of [25], the conductor dimension of a family of elliptic curves ordered by heights is also (non-conjecturally) $\frac{5}{6}$, so we may need further local averaging over $X^{1/6+\epsilon}$ many primes. The local averaging introduced in Theorem 5.1 is slightly different from Sarnak's suggestion, since they take a local average over $\Theta(N_E)$-many primes, not $O(H_E^{1/6+\epsilon})$.

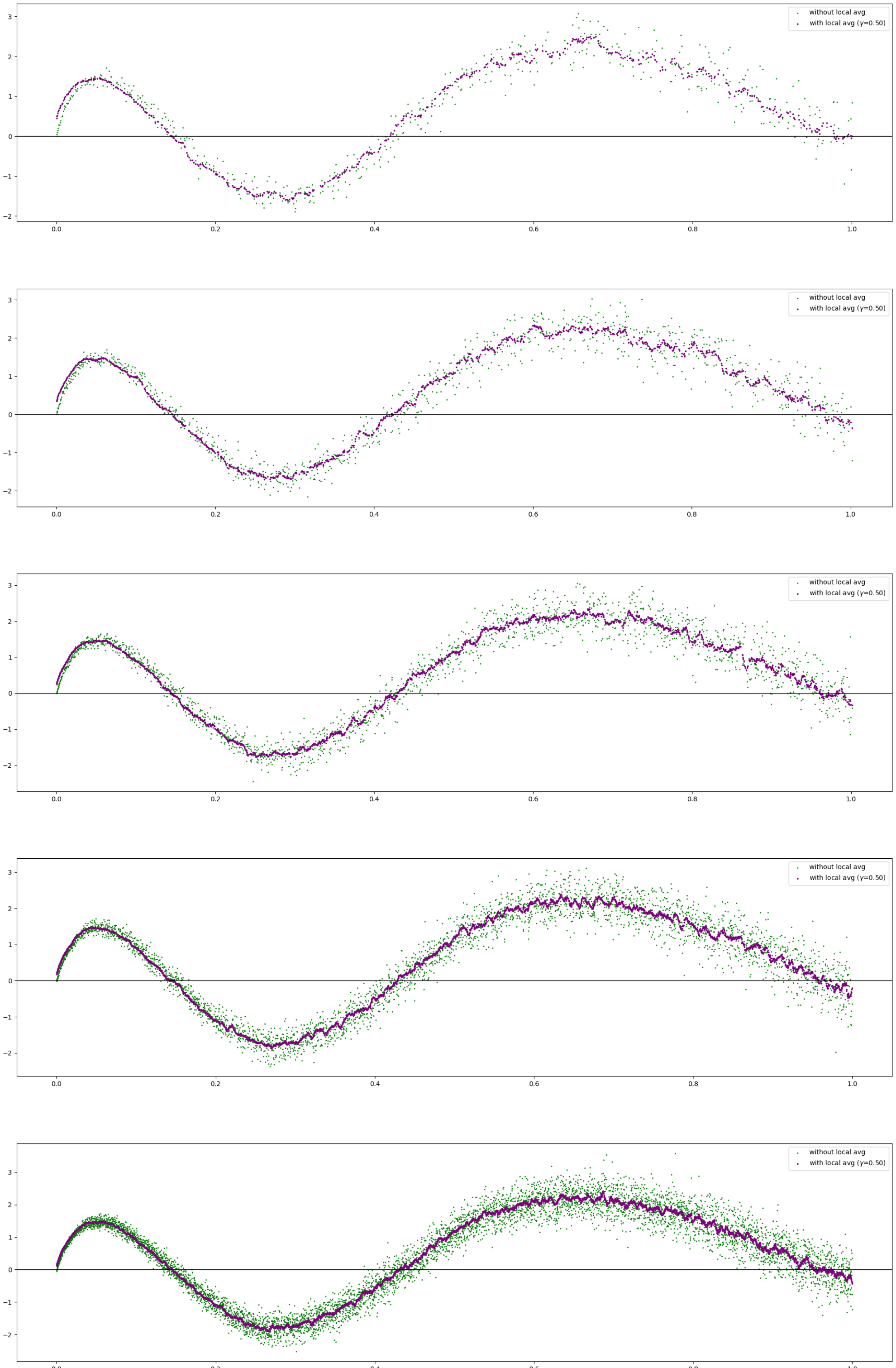


Figure 8: Murmuration of non-CM elliptic curves with conductor in $[2^k, 2^{k+1})$ and primes $p < 2^k$ for $k = 12, \ldots, 16$ weighted by root numbers, without (resp. with) local averaging of $X^{\gamma} = X^{\frac{1}{2}}$-many primes.

7.4 Hecke characters of imaginary quadratic fields

For each $D \equiv 3 \pmod 4$, the number of Hecke characters of $\mathrm{Cl}(\mathbb{Q}(\sqrt{-D}))$ is $h(-D)$, and the class number formula

$$h(-D) = \frac{\sqrt{D}}{\pi} L(1, \chi_{-D})$$

with the estimates $|D|^{-\epsilon} \ll L(1, \chi_{-D}) \ll |D|^{\epsilon}$[4] implies that the conductor dimension of $\mathcal{F}$ is $\frac{3}{2}$. In particular, it is larger than 1, so we may not need any local averaging from (34). However, we still include it in Theorem 6.2 to remove the dependence on p in the main term of Theorem 6.1. The main term

$$M(y) = \overline{c} \sum_{1 \le m < 2\sqrt{y}} \overline{M}_m(y) + M_-(y)$$

in Theorem 6.2 is the murmuration density for the family $\mathcal{F}$, so we expect that

$$\mathbb{E}_{P \le p \le P+N} \left[\mathbb{E}_{\psi \in \mathcal{F}} [\sqrt{p} a_\psi(p); \Phi, N] \right] = M_\Phi \left(\frac{p}{N} \right) + o(1) \tag{47}$$

for a compactly supported smooth function Φ on $(0, \infty)$ and

$$M_\Phi(y) = \frac{\int_0^\infty M\left(\frac{y}{u}\right) \Phi(u) u^{3/2} \frac{du}{u}}{\int_0^\infty \Phi(u) u^{3/2} \frac{du}{u}}. \tag{48}$$

The family is expected to have a symplectic symmetry type [9, 24], so we expect

$$\lim_{y \to 0^+} M_\Phi(y) = 0, \quad \lim_{y \to \infty} M_\Phi(y) = -\frac{1}{2}$$

and this is Theorem 3 of [36]. (Recall that Zubrilina's family has an orthogonal symmetry type.)

[4] The lower bound is Siegel's theorem, while the upper bound follows from the Pólya–Vinogradov inequality.

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