Unramified extension of $\mathbb{Q}(\sqrt{3})$

Seewoo Lee

February 15, 2022

In this note, we show that there's no unramified extension of $\mathbb{Q}(\sqrt{3})$. Before we start, let's recall the easier case - $\mathbb{Q}$. If $K/\mathbb{Q}$ is a finite extension, then the Minkowski's bound tell's us that for any ideal class $A \in \mathrm{Cl}_K$, there's a nonzero integral ideal $\mathfrak{a} \subseteq \mathcal{O}_K$ in A such that

$$[\mathcal{O}_K : \mathfrak{a}] = \mathcal{N}(\mathfrak{a}) \leq \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s \sqrt{|d_K|}$$

where $n = [K : \mathbb{Q}]$, s is the number of (pairs of) complex places, and d_K is the discriminant of K. We know that the prime $p \in \mathbb{Q}$ ramifies in K if and only if $p|d_K$. Since $\mathcal{N}(\mathfrak{a}) \geq 1$, we have

$$|d_K| \geq \frac{n^n}{n!}\left(\frac{\pi}{4}\right)^s \geq \frac{n^n}{n!}\left(\frac{\pi}{4}\right)^{n/2}$$

If we define RHS as a_n, then

$$\frac{a_{n+1}}{a_n} = \left(\frac{\pi}{4}\right)^{1/2}\left(1 + \frac{1}{n}\right)^n \geq 2\sqrt{\frac{\pi}{4}} = \sqrt{\pi} > 1$$

and $a_2 = \frac{\pi}{2} > 1$, so $a_n > 1$ for all $n \geq 1$. This implies that for any nontrivial extension K of $\mathbb{Q}$, $|d_K| > 1$ so there exists a prime $p \in \mathbb{Q}$ that divides d_K, so that ramifies in K.

To obtain the similar result for $\mathbb{Q}(\sqrt{3})$ or any other number fields, we may need the (global) class field theory. According to the class field theory, for any number field K, there exists the *Hilbert class field* H_K, which is a maximal unramified finite abelian extension of K and $\mathrm{Gal}(H_K/K) \simeq \mathrm{Cl}_K$ canonically (via Artin reciprocity map). So if K has a class number 1 (i.e. $\mathcal{O}_K$ is a PID), then there's no nontrivial unramified *abelian* extension of K. (Here unramifiedness includes archimedean places. For example, $L/\mathbb{Q}(\sqrt{3})$ is unramified at real places if and only if L is a totally real field.)

But how to show that there's no unramified extension including non-abelian ones? For a number field extension $M/L/K$, the *relative* discriminant satisfy the relation

$$\Delta_{M/K} = \mathcal{N}_{L/K}(\Delta_{M/L})\Delta_{L/K}^{[M:L]}$$

where $\mathcal{N}_{L/K} : I_L \to I_K$ is the ideal norm map. Now assume that there's a nontrivial unramified extension K of $\mathbb{Q}(\sqrt{3})$. By applying the above relation, we get

$$\Delta_{K/\mathbb{Q}} = \Delta_{\mathbb{Q}(\sqrt{3})/\mathbb{Q}}^{[K:\mathbb{Q}\sqrt{3}]} = 12^n$$

where $n = [K : \mathbb{Q}(\sqrt{3})]$. ($\Delta_{K/\mathbb{Q}(\sqrt{3})} = (1)$ since $K/\mathbb{Q}(\sqrt{3})$ is unramified.) Now the Minkowski's bound gives

$$\frac{(2n)!}{(2n)^{2n}} \cdot 12^{n/2} \geq 1.$$

(We have $s = 0$ since we are assuming that archimedean places also unramifies.) We can show that this inequality fails for big n. In fact, if we put LHS as b_n then

$$\frac{b_{n+1}}{b_n} = \left(1 + \frac{1}{n}\right)^{-2n} \frac{2n+1}{2n+2}\sqrt{12} \leq \frac{\sqrt{12}}{4} \leq 1$$

for any $n \geq 1$, and $a_3 < 1$. So we get $n \leq 2$, and we already know that there's no degree 2 unramified extension of $\mathbb{Q}(\sqrt{3})$ because every degree 2 extension is abelian!

What if we allow infinite places to be ramify? Then there's such extension. We will show that $K = \mathbb{Q}(\sqrt{3}, \sqrt{-1})$ is such extension. First, since $d_{\mathbb{Q}(\sqrt{-1})} = -4$, the only prime $p \in \mathbb{Q}$ ramifies in $\mathbb{Q}(\sqrt{-1})$ is 2. So if $p \neq 2$, then p is unramified in $\mathbb{Q}(\sqrt{-1})$, and this implies that any prime $\mathfrak{p}|p$ in $\mathbb{Q}(\sqrt{3})$ is unramified in $K = \mathbb{Q}(\sqrt{3}, \sqrt{-1})$. For $p = 2$, assume that the prime $\mathfrak{p}$ lying over 2 ramifies in K. Then the ramification degree of 2 in K is 4 since 2 also ramifies in $\mathbb{Q}(\sqrt{3})$. However, this is impossible since 2 does not ramify in the subfield $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$, which has a discriminant $d_{\mathbb{Q}(\sqrt{-3})} = -3$. Hence any finite prime in $\mathbb{Q}(\sqrt{3}$ is unramified in K. But the infinite place ramifies in K since K has a complex place (K is not a totally real field).