Heisenberg's principle and positive functions

Principe d'Heisenberg et fonctions positives

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Abstract

We consider a natural problem concerning Fourier transforms. In one variable, one seeks functions f and $\widehat{f}$, both positive for $|x| \geq a$ and vanishing at 0. What is the lowest bound for a? In higher dimension, the same problem can be posed by replacing the interval by the ball of radius a. We show that there is indeed a strictly positive lower bound, which is estimated as a function of the dimension. In the last section the question, and its solution, are shown to be naturally related to the theory of zeta functions.

Introduction

Heisenberg's inequalities are expressed, in terms of the notations of the present article, as

$$\int x^2 |f(x)|^2 \mathrm{d}x \int y^2 |\widehat{f}(y)|^2 \mathrm{d}y \geq \frac{1}{16\pi^2}$$

(if f has norm 1), and they are optimal, since the equality holds for $f(x) = e^{-\pi x^2}$. In the following form

$$\Delta p \Delta x \geq \hbar$$

they are interpreted by physicists as a relationship between the standard deviation of position and momentum of an object; when these relations appeared,

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they are called as "uncertainty relations", since it is a question about determining the position and the momentum of a point mass precisely. For mathematicians, it is about a simple observation, but with different aspects: for a Foutier transform pair of functions, one cannot make both to be concentrated near 0. This common but important fact is called Heisenberg's principle.

In this article, we introduce an another problem about positivity of functions outside a neighborhood of zero. This would not give anything new if we do not impose a new condition, for both functions, to be negative at zero. Can the neighborhoods of zero where the functions are positive outside of those be arbitrarily small? We will see that the answer is negative.

The problem, and the first answers for higher dimensions, comes from number theory, more precisely from Tate's theory of zeta functions of number fields. The functional equation of adélic zeta functions poses the problem in the most natural way, as we show in §4 of this article; however, it is already implicitly proposed in the classical article of Landau [2]. But the classical Fourier analysis gives the best results, as shown in the first three sections. The results are about lower and upper bounds of natural constants associated to the problem, denoted as B_d and $\mathcal{B}_d$ in terms of the dimension d. They are far from being optimal. Section 1 and 2 treats upper and lower bounds for $d = 1$; section 3 for $d \geq 2$.

Finally, the section 4, *Arithmetic*, using Tate's method, relates this problem to the study of real zeros of zeta functions in relation to the discriminant. The arithmetic argument shows that the linear growth of B_d as a function in the dimension is natural in view of known ramification properties of these fields.

1 Statement of the problem and lower bound of B_1

Consider a pair of functions $(f, \widehat{f})$ on the real line: it is a Fourier pair if

$$\begin{cases} \widehat{f}(y) = \int f(x) e^{-2i\pi xy} \mathrm{d}x, & f \in L^1(\mathbb{R}) \\ f(x) = \int \widehat{f}(y) e^{2i\pi xy} \mathrm{d}y, & \widehat{f} \in L^1(\mathbb{R}). \end{cases}$$

So f and $\widehat{f}$ are continuous and converges to 0 at infinity. We are interested in the Fourier pairs $(f, \widehat{f})$ such that

1. f and $\widehat{f}$ are real-valued, even, and not identically zero,
2. $f(0) \leq 0$ and $\widehat{f}(0) \leq 0$,

3. $f(x) \geq 0$ for $x \geq a_f$ and $\widehat{f}(y) \geq 0$ for $y \geq a_{\widehat{f}}$.

Note that the condition 2 and the non-vanishing assumptions on f and $\widehat{f}$ imply a_f and $a_{\widehat{f}} > 0$.

Problem. What is the infimum of the product $a_f a_{\widehat{f}}$ for the Fourier pairs $(f, \widehat{f})$ satisfying 1–3?

We denote the infimum as $B_1 \geq 0$ (note that the pair attaining infimum clearly exists). We will show, which is not obvious a priori, that B_1 is strictly positive.

Until section 3, we will focus on dimension 1. For a Fourier pair $(f, \widehat{f})$ satisfying 1–3 let

$$A(f) = \inf\{x > 0 : f((x, \infty)) \subset \mathbb{R}^+\}$$
$$A(\widehat{f}) = \inf\{y > 0 : \widehat{f}((t, \infty)) \subset \mathbb{R}^+\}.$$

The product $A(f)A(\widehat{f})$ is invariant under scaling, i.e. replacing $f(x)$, $\widehat{f}(y)$ by $f(x/\lambda)$, $\lambda\widehat{f}(\lambda y)$, $\lambda > 0$. Since

$$B_1 = \inf A(f)A(\widehat{f})$$

for all Fourier pairs satisfying 1–3, we limit ourselves to those which $A(f) = A(\widehat{f})$. Then $f + \widehat{f} \neq 0$ (consider their values at points near $A(f)$ and greater than it), and

$$A(f + \widehat{f}) \leq A(f) = A(\widehat{f}).$$

So $B_1 = \inf A^2(f + \widehat{f})$. Hence we see that

$$B_1 = A^2, \quad A = \inf A(f)$$

where the infimum is taken over all functions $f \in L^1(\mathbb{R})$, real-valued and even, not identically zero, equal to their own Fourier transforms, and $f(0) < 0$.

Let

$$\gamma(x) = e^{-\pi x^2}$$

so that $\gamma = \widehat{\gamma}$. If $f(0) < 0$, $f - f(0)\gamma$ satisfies the same conditions as f, and

$$A(f - f(0)\gamma) \leq A(f).$$

As a result,

$$A = \inf A(f) \tag{1.1}$$

where the infimum is taken over all $f \in L^1(\mathbb{R})$, real-valued, even, not identically zero, $f = \widehat{f}$, and $f(0) = 0$.

Here is an important result.

Theorem 1.1. Let $\lambda = -\inf\left(\frac{\sin x}{x}\right) = 0.2712\cdots$. Then

$$A \geq \frac{1}{2(1+\lambda)} = 0.4107\cdots$$

and so

$$B \geq 0.1687\cdots.$$

Proof. Choose $f = \widehat{f}$, $f(0) = 0$, and $\int_{\mathbb{R}} |f(x)|\mathrm{d}x := \int_{\mathbb{R}} |f| = 1$. Write $A = A(f)$. Put $f = f^+ - f^-$, $|f| = f^+ + f^-$. Since $\int_{\mathbb{R}} f = \widehat{f}(0) = 0$, we have $\int_{\mathbb{R}} f^+ = \int_{\mathbb{R}} f^- = \int_{-A}^{A} f^- = \frac{1}{2}$. So $\int_{-A}^{A} |f| \geq \frac{1}{2}$. From $|f(x)| \leq \int |\widehat{f}| = 1$, $2A \geq \frac{1}{2}$ and we obtain the first bound $A \geq \frac{1}{4}$. We will see that this argument generalizes to higher dimensions.

In dimension 1, we can refine it in the following way. From $f = \widehat{f}$,

$$\begin{aligned} f(x) &= \int f(y)\cos 2\pi yx\mathrm{d}y = \int f(y)(\cos 2\pi yx - 1)\mathrm{d}y \\ &= \int f^-(y)(1-\cos 2\pi yx)\mathrm{d}y - \int f^+(y)(\cos 2\pi yx - 1)\mathrm{d}y. \end{aligned}$$

This implies, both integrals being positive,

$$f^-(x) \leq \int f^+(y)(1-\cos 2\pi yx)\mathrm{d}y$$

and

$$\frac{1}{4} = \int_0^A f^- \leq \int_0^\infty f^+(y)\left(A - \frac{\sin 2\pi yA}{2\pi y}\right)\mathrm{d}y$$

so

$$\frac{1}{4} \leq \frac{A}{2}\sup_{u\in\mathbb{R}}\left(1-\frac{\sin u}{u}\right) = \frac{A}{2}(1+\lambda)$$

and we obtain the theorem. □

Later, we will need to consider functions that are regular enough. A natural class is the Schwartz space $\mathcal{S}$. It is not obvious that the infimum A defined by (1.1), taken only over the functions in $\mathcal{S}$, coincides with that over all $f \in L^1(\mathbb{R})$.

Let $\mathcal{B}_1$ be A^2, where A is defined by (1.1) for $f \in \mathcal{S}$. We will see that B_1 and $\mathcal{B}_1$ are not much different. Clearly, we have

$$B_1 \leq \mathcal{B}_1. \tag{1.2}$$

Let

$$B_1^- = \inf\{A^2 : f(0) < 0, f = \widehat{f} \text{ even} \neq 0, f \in L^1(\mathbb{R})\}.$$

Hence B_1^- is defined by (1.1), where we impose $f(0) < 0$. Define $\mathcal{B}_1^-$ similarly for $f \in \mathcal{S}$. Clearly,

$$B_1^- \leq \mathcal{B}_1^- \tag{1.3}$$

$$\mathcal{B}_1 \leq \mathcal{B}_1^-, \quad B_1 \leq B_1^-. \tag{1.4}$$

To prove $\mathcal{B}_1^- \leq B_1^-$, let $f \in L^1(\mathbb{R})$ be a function satisfying the conditions for (1.1) but $f(0) < 0$, and let $a = A(f)$. Let $\varphi = \psi * \psi$, where ψ is C^∞, even, positive, and compactly supported near 0, and $g = f * \varphi$. Then $A(g) \leq a + \varepsilon$ and $g(0) < 0$. We have $\widehat{g} = \widehat{f}\widehat{\psi}^2$; by applying the same operation on $\widehat{g}$ we obtain a function $h \in \mathcal{S}$ such that $h = \widehat{h}$, $h(0) < 0$, and $A(h) \leq a + \varepsilon$; from this we get $\mathcal{B}_1^- \leq B_1^-$ and

$$\mathcal{B}_1^- = B_1^-. \tag{1.5}$$

Note that the argument fails when $f(0) = 0$. We will show

$$B_1^- \leq 2B_1; \tag{1.6}$$

combining (1.4) and (1.6) gives

$$B_1 \leq \mathcal{B}_1 \leq 2B_1. \tag{1.7}$$

Let f be a function satisfying the conditions for (1.1) and $a = A(f)$. Since $\widehat{f}(0) = \int f(x)\mathrm{d}x = 0$, f takes a negative value on $[-a, a]$. Let $b > 0$ be such a number, and consider the distribution

$$T = \delta_b + \delta_{-b} + 2\delta_0.$$

It is a positive measure with nonnegative Fourier transform

$$\widehat{T} = 2\cos(2\pi b y) + 2 \geq 0.$$

We have

$$(T * f)(0) = f(b) + f(-b) < 0.$$

Since $b < a$, $g = T * f$ satisfies

$$g(0) < 0, \quad g \geq 0 \text{ on } (2a, \infty).$$

Moreover $\widehat{g} = \widehat{T}\widehat{f}$ is nonnegative on $[0, \infty)$, and $\widehat{g}(0) = 0$. By scaling, we obtain a function h such that

$$h \geq 0 \text{ on } [a\sqrt{2}, \infty), \quad h(0) < 0$$
$$\widehat{h} \geq 0 \text{ on } [a\sqrt{2}, \infty), \quad \widehat{h}(0) = 0.$$

The functions h and $\widehat{h}$ are real-valued and even. Hence $h + \widehat{h}$ satisfy the conditions defining B_1^-. So $B_1^- \leq (a\sqrt{2})^2 = 2a^2$; by varying f, we obtain (1.6).

2 Upper bound of B_1

A first idea is to associate f with the Hermite series

$$f(x) \sim \sum_{n=0}^{\infty} a_n h_n(x)$$

where h_n are eigenvectors of the Fourier transform $\mathcal{F}$ corresponding to the eigenvalues i^n. Since $f = \widehat{f}$ the expression becomes

$$f(x) \sim \sum_{m=0}^{\infty} a_{4m} h_{4m}(x).$$

Each h_n has a form of $h_n = e^{-\pi x^2} P_n(x)$ where P_n is a polynomial of degree n. A suitable linear combination of h_0 and h_4 (satisfying $f(0) = 0$) gives $\pi A^2 \leq 3$. The calculations seem difficult and we will not proceed in this direction further.

We can also consider the functions

$$g_a(x) = a\gamma(ax) + \gamma\left(\frac{x}{a}\right) - (1+a)\gamma(x), \quad a > 1 \tag{2.1}$$

which satisfy the requirements for (1.1). Then any expression of the form

$$\int_1^{\infty} g_a(x) \mathrm{d}\tau(a) \tag{2.2}$$

where τ is a measure on $[1, \infty)$ such that the integral converges absolutely and ≥ 0 at infinity is our candidate (it seems difficult to characterize such measures where (2.2) converges absolutely and positive at infinity).

We first study $A(g_a)$. It is convenient to put $X = \pi x^2$, and $G_a(X) = g_a(x)$, so

$$G_a(X) = ae^{-a^2 X} + e^{-a^{-2} X} - (1+a)e^{-X}.$$

The function

$$H_a(X) = e^X G_a(X) = ae^{(1-a^2)X} + e^{(1-a^{-2})X} - 1 - a \tag{2.3}$$

is convex and satisfies

$$H_a(0) = 0, \quad H_a'(0) = -a^2(a^2-1)(a^3-1) < 0$$

and tends to $+\infty$ as $X \to \pm\infty$. So it has a unique zero $X_a > 0$, and

$$A(g_a) = \sqrt{\frac{X_a}{\pi}}.$$

Upper bound of B_1

It is natural to study how X_a varies, and we first consider those for a close to 1. Put $a = 1 + h$, $h > 0$, then for fixed X it can be written as

$$H_a(X) = (1+h)(e^{-X(2h+h^2)} - 1) + e^{X(2h-3h^2+3h^3-4h^4)X} - 1$$

modulo $O(h^5)$. It can be written as $P_1 h + P_2 h^2 + P_3 h^3 + P_4 h^4 + O(h^5)$, where the polynomials P_i are

$$\begin{aligned} P_1 &= 0 \\ P_2 &= 2X(2X-3) \\ P_3 &= -X(2X-3) \\ P_4 &= -5X + 15X^2 - \frac{28}{3}X^3 + \frac{4}{3}X^4. \end{aligned}$$

From the expression of P_2, for sufficiently small h, $H_a(X) > 0$ if $X > \frac{3}{2}$ and $H_a(X) < 0$ if $X < \frac{3}{2}$. As a result,

$$\lim_{a\to 1^+} X_a = \frac{3}{2}. \tag{2.4}$$

which gives an explicit bound

$$A \le \sqrt{\frac{3}{2\pi}}. \tag{2.5}$$

But this simple bound cannot be the true value of A. For $X = \frac{3}{2}$, P_2 and P_3 cancel out, and

$$P_4\left(\frac{3}{2}\right) = \frac{3}{2}.$$

For nonzero small h, we therefore have $X_a < \frac{3}{2}$.

If $a \to +\infty$, $X_a \to +\infty$; in fact, a simple calculation shows that

$$X_a = \log a + O(1) \quad (a \to +\infty).$$

We have not determined the minimum value of X_a, but it is easy to estimate it semi-heuristically. The value $a = \sqrt{2}$ satisfies, for $q = e^{\frac{1}{2}X_a}$,

$$q^3 - (1+\sqrt{2})q^2 + \sqrt{2} = 0;$$

if $q \neq 1$, it becomes the quadratic equation

$$q^2 - \sqrt{2}q - \sqrt{2} = 0$$

with a zero $q = \frac{\sqrt{2}}{2}(1 + \sqrt{1 + 2\sqrt{2}})$,

$$X_a = 2\log q = 1.4749\cdots < \frac{3}{2} \quad (a = \sqrt{2}).$$

The value $a = 2$ gives, for $q = e^{\frac{3}{4}X}$,

$$q^4 - 2\frac{q^4 - 1}{q - 1} = 0.$$

The unique zero $q > 1$ is $q = 2.9744\cdots$, from where

$$X_a = 1.4534\cdots \quad (a = 2).$$

It seems that we can approximate the optimal value by this method. Indeed, if we solve $H_a(X) = 0$ for H_a given by (2.3), and if we assume $a \geq 2$, the first term is negligible. So X_a is approximately

$$\frac{\log(1 + a)}{1 - a^{-2}}.$$

The extremal value of this expression is attained when $a(1 - a) = 2\log(1 + a)$, which gives

$$a = 2.08137\cdots.$$

In all cases, the minimum value of $A(g_a)$ we obtain is not the optimum of (1.1) that we are looking for. Consider a_0 such that $X_0 = X_{a_0}$ is minimal, and $H_0 = H_{a_0}$ is positive on $[X_a, \infty)$. Let a be a number (for example, near 1) such that $X_a > X_0$. On $[X_a, \infty)$, $H_a \geq 0$ and its order of growth as $X \to +\infty$, $e^{(1-a^{-2})X}$, is smaller than that of H_{a_0} if $a < a_0$. So there exists $T > 0$ such that $H_{a_0} - TH_a$ is ≥ 0 on $[X_a, \infty)$. But this function is positive on $[X_0, X_a)$, so is for $X \geq X'$ with $X' < X_0$.

The same argument holds for all a_0 with $X_0 < \frac{3}{2}$. For $a_0 = 2$, we can determin the optimal value (corresponds to a very close to 1), giving a function ≥ 0 on $[X'', \infty)$ where

$$\begin{aligned} X'' &= 1.25\cdots \\ A &\leq 0.63\cdots. \end{aligned} \tag{2.6}$$

We only made a very rough calculation. Nevertheless we state the result, to compare with Theorem 1.1.

Theorem 2.1. We have $A \leq 0.64\cdots$ and $B_1 \leq 0.41\cdots$.

3 Higher dimensions

On Euclidean space $\mathbb{R}^d$ with inner product

$$x \cdot y = \sum_{i=1}^{d} x_i y_i, \quad \|x\| = (x \cdot x)^{1/2},$$

Fourier transform is defined by

$$\widehat{f}(y) = \int f(x) e^{-2i\pi x \cdot y} \mathrm{d}x \tag{3.1}$$

where $\mathrm{d}x = \mathrm{d}x_1 \cdots \mathrm{d}x_d$ is the Lebesgue measure; so

$$f(x) = \int \widehat{f}(y) e^{2i\pi x \cdot y} \mathrm{d}y. \tag{3.2}$$

We suppose that f and $\widehat{f}$ are continuous and integrable. More generally, if E is a Euclidean space of dimension d, if the invariant measure $\mathrm{d}x$ on E is chosen so that the cube formed by the orthonormal basis has measure 1, and if $x \cdot y$ is the inner product, Fourier transform and its inverse are defined by (3.1) and (3.3).

Consider the Fourier pairs $(f, \widehat{f})$ satisfying

1. $f, \widehat{f}$ are not identically zero,
2. $f(0) \leq 0$ and $\widehat{f}(0) \leq 0$, (3.3)
3. $f(x) \geq 0$ for $\|x\| \geq a_f$, $\widehat{f}(0) \geq 0$ for $\|y\| \geq a_{\widehat{f}}$.

Define $A(f)$ and $A(\widehat{f})$ as in §1:

$$A(f) = \inf\{r > 0 : f(x) \geq 0 \text{ if } \|x\| > r\},$$

and

$$B_d = \inf A(f) A(\widehat{f})$$

for pairs satisfying 1–3. Let $f^\sharp(x)$ be the (invariant) integral of f on the sphere of radius $\|x\|$: $\widehat{f^\sharp} = (\widehat{f})^\sharp$ and $f^\sharp$ and $\widehat{f^\sharp}$ are nonzero; otherwise f and $\widehat{f}$ are compactly supported from 3. Since $A(f^\sharp) \leq A(f)$ and $A(\widehat{f^\sharp}) \leq A(\widehat{f})$, we can limit ourselves to the radial functions. Since

$$(f(x/\lambda))^\wedge = \lambda^d \widehat{f}(\lambda y) \quad (\lambda > 0),$$

we can follow the argument in §1 and we have

$$B_d = A^2, \quad A = \inf A(f) \tag{3.4}$$

where the infimum is taken over the functions $f \in L^1(\mathbb{R}^d)$, radial, not identically zero, such that $f = \widehat{f}$ and $f(0) = 0$.

We can, as in §1, if necessary, add a multiple of the following radial and self-dual function

$$\gamma(x) = e^{-\pi\|x\|^2}.$$

Theorem 3.1. We have

$$B_d \geq \frac{1}{\pi}\left(\frac{1}{2}\Gamma\left(\frac{d}{2}+1\right)\right)^{2/d} > \frac{d}{2\pi e}.$$

Proof. Follow the argument of the case $d = 1$, where we replace the interval $(-A(f), A(f))$ with the ball of radius $A(f)$ centered at the origin, whose volume $(\geq \frac{1}{2})$ is $\frac{1}{\Gamma(\frac{d}{2}+1)}(A(f))^d \pi^{d/2}$. □

Put $X = \pi\|x\|^2$, the argument in §2 natually leads us to consider the functions

$$g_a(x) = G_a(X) \quad (x \in \mathbb{R}^d)$$

where

$$G_a(X) = a^d e^{-Xa^2} + e^{-Xa^2} - (1 + a^d)e^{-X},$$

and set

$$H_a(X) = a^d e^{(1-a^2)X} + e^{(1-a^{-2})X} - (1 + a^d), \quad a > 1.$$

It is convenient to define $a^2 = 1 + k$, $d = 2c$, which gives

$$H_a(X) = (1+k)^c e^{-kX} + e^{(1-(1+k)^{-1})X} - 1 - (1+k)^c.$$

The derivative in X at the origin is

$$\frac{k}{1+k}\left(1 - (1+k)^{c+1}\right) < 0;$$

the convexity argument in §2 shows that H_a has a unique positive zero X_a. As before, we compute the expansion of $H_a(X)$ in k up to order 4. It becomes

$$H_a(X) = P_1 k + P_2 k^2 + P_3 k^3 + P_4 k^4 + O(k^5)$$

where

$$P_1 = 0$$
$$P_2 = X(X - c - 1)$$
$$P_3 = \frac{1}{2}(c-2)X(X-c-1)$$

Higher dimensions

$$P_4 = \frac{1}{12}X(X^3 - (2c+6)X^2 + (3c(c-1)+18)X - (2c(c-1)(c-2)+12)).$$

As in the dimension 1 case, we see that P_2 and P_3 cancel out for

$$X = X(d) := \frac{d}{2} + 1. \tag{3.5}$$

Moreover, $P_2 > 0$ for $X > X(d)$, < 0 for $X < X(d)$. Taking the limit $k \to 0$ gives

$$\lim_{a\to 1} X_a = \frac{d}{2} + 1.$$

To understand the location of X_a with respect to $X(d)$ as $a \to 1$, we compute $Q_4(X(d))$ or $P_4 = \frac{X}{12}Q_4$. Calculation gives

$$Q_4(c+1) = -c^2 + 1.$$

For $d > 2$, this term is < 0, so $H_a(X(d)) < 0$ for a close to 1, which shows that

$$X_a > \frac{d}{2} + 1 \quad (a > 1,\ \text{close to } 1).$$

Therefore it is possible that the value (3.5) is optimal. This is not the case when $d = 1$ as we saw in §2.

For $d = 2$, $Q_4(c+1) = 0$, so we need to compute up to degree 5, where

$$H_a(2) = (1+k)e^{-2k} + e^{2(1-\frac{1}{1+k})} - 2 - k. \tag{3.6}$$

The Taylor series at 0 of

$$f(z) = e^{2(1-\frac{1}{1+z})} = e^{2\frac{z}{1+z}},$$
$$f(z) = \sum_{n=0}^{\infty} q_n z^n,$$

can be calculated by the residue theorem. Let

$$w = \frac{z}{1+z},\ z = \frac{w}{1-w},\ \mathrm{d}z = \frac{\mathrm{d}w}{(1-w)^2},$$

by taking a small contour around 0:

$$\begin{aligned} q_n &= \mathrm{Res}_{z=0}\frac{f(z)}{z^{n+1}} = \frac{1}{2i\pi}\oint e^{\frac{2z}{1+z}}\frac{\mathrm{d}z}{z^{n+1}} \\ &= \frac{1}{2i\pi}\oint e^{2w}\frac{(1-w)^{n+1}}{w^{n+1}}\frac{\mathrm{d}w}{(1-w)^2} \\ &= \mathrm{Res}_{w=0}\frac{(1-w)^{n-1}}{w^{n+1}}e^{2w}. \end{aligned}$$

In particular, q_5 is the sum of

$$\frac{2^4}{4!} - \frac{2^5}{5!} \tag{3.7}$$

coming from the first term of (3.6), and the coefficient of w^5 in $e^{2w}(1-w)^4$, equal to

$$\frac{2^5}{5!} - 4 \cdot \frac{2^4}{4!} + 6 \cdot \frac{2^3}{3!} - 4 \cdot \frac{2^2}{2!} + 2. \tag{3.8}$$

We find that $q_5 = 0$.

Similarly, q_6 is the sum of

$$-\frac{2^5}{5!} + \frac{2^6}{6!} \tag{3.9}$$

and

$$\frac{2^6}{6!} - 5 \cdot \frac{2^5}{5!} + 10 \cdot \frac{2^4}{4!} - 10 \cdot \frac{2^3}{3!} + 5 \cdot \frac{2^2}{2!} - 2, \tag{3.10}$$

which is

$$q_6 = -\frac{4}{45} < 0.$$

When a is sufficiently close to 1, we therefore have $H_a(2) < 0$ and $X_a > X(2) = 2$. Again, the bound given by (3.5) could be optimal.

To conclude this section, note that for all $d \geq 2$ we obtain the upper bound

$$B_d \leq \mathcal{B}_d \leq \frac{d+2}{2\pi} \tag{3.11}$$

where $\mathcal{B}_d$ is defined, as in §1, by the functions in the space $\mathcal{S}(\mathbb{R}^d)$. Also following the argument in the end of §1, relating the bounds for L^1 and $\mathcal{S}$ applies. To prove the inequality (1.6), we have to consider $T = \delta_b + \delta_{-b} + 2\delta_0$, where $\|b\| < a = A(f)$ and $f(b) < 0$; $\widehat{T} = 2\cos(2\pi b \cdot y) + 2$ is a positive plane wave. The rest of the argument is the same, replacing $h + \widehat{h}$ with the spherical average of $h + \widehat{h}$ if we want to stick to the radial functions. In conclusion, comparing with Theorem 3.1,

Theorem 3.2. We have

$$B_d \leq \mathcal{B}_d \leq \frac{d+2}{2\pi}, \quad B_d \geq \frac{1}{2}\mathcal{B}_d. \tag{3.12}$$

4 Arithmetic

Let F be a number field of degree d over $\mathbb{Q}$. We denote as v for the places of F (finite or archimedean), and F_v for the corresponding completion; for finite v,

$O_v \subset F_v$ is the ring of integers of F_v and $O_v^\times$ is the group of unities; q_v is the cardinality of the residue field. Let

$$\mathbb{A}_F = \prod_v{}' F_v$$

(restricted product) be the ring of adèles of F, and $\mathbb{A}_F^\times = I_F$ the group of idèles. Let $x : I_F \mapsto \prod_v |x|_v$ be the idèle norm,

$$I_F^1 = \{x \in I_F : |x| = 1\}$$
$$\text{and } I_F^+ = \{x \in I_F : |x| \geq 1\}.$$

Consider the invariant measure $\mathrm{d}x = \prod \mathrm{d}x_v$ on $\mathbb{A}_F$, where $\mathrm{d}x_v$ is Haar measure on F_v. At finite places, $\mathrm{d}x_v$ are self-dual measures of Tate [6]; at a real place, $\mathrm{d}x_v$ is the Lebesgue measure; at a complex place, if we write the variable $z = x + iy$, $\mathrm{d}z = 2\mathrm{d}x\mathrm{d}y$. At a real place, the Fourier transform $\hat{f}(y)$ of a function f is defined as before.

If $z = x + iy$ is a complex variable and $w = \xi + i\eta$, Tate define the transfom $\widehat{f}(w)$ of a function $f(z)$ by

$$\widehat{f}(w) = \int f(z) e^{-2i\pi \mathrm{Tr}(zw)} \mathrm{d}z$$
$$\text{where } \mathrm{Tr}(zw) = 2\Re(zw) = 2(x\xi - y\eta).$$

For radial functions, in each of the variables, it coincides with the Fourier transform defined in §3 with the inner product $z \cdot w = 2(x\xi + y\eta)$. The self-dual measure $\mathrm{d}z$ of Tate is the normalized measure considered in the beginning of §3 for abstract Euclidean spaces.

Let f be a function in the Schwartz space of $\mathbb{A}_F$ given by

$$f(x) = \prod_{v|\infty} f_v(x_v) \prod_{v \text{ finite}} f_v^0(x_v) \tag{4.1}$$

where f_v^0 is the characteristic function of O_v and, for archimedean v, f_v is for a moment an arbitrary Schwartz function. Tate's zeta function associated to f is defined for $\Re(s) > 1$ by

$$Z(f, s) = \int_{I_F} f(x) |x|^s \mathrm{d}^\times x,$$

where $\mathrm{d}^\times x$ is the product of $\mathrm{d}^\times x_v = \frac{\mathrm{d}x_v}{|x_v|}$ (multiplied by $(1 - q_v^{-1})^{-1}$ at finite places).

Rather than the factorizable functions in (4.1), we will consider the functions of the form $g_a(x)$ (§3) on $\mathbb{R}^d$, where $\mathbb{R}^d$ is regarded as an inner product space via

$$\|x_\infty\|^2 = \sum_{v \text{ real}} |x_v|^2 + \sum_{v \text{ complex}} 2\|z_v\|^2$$

where $\|z\|$ is the usual absolute value of a complex number (We denote $|z| = \|z\|^2$ the normalized absolute norm as in Tate's theory). More generally,

$$f(x) = f_\infty(x_\infty) \prod_{v \text{ finite}} f_v^0(x_v) \tag{4.2}$$

where $f_\infty(x_\infty) \in \mathcal{S}(\mathbb{R}^d)$. The conditions imposed by Tate (i.e., (z_1), (z_2), (z_3) in [6, §4.4]) are satisfied by these functions. For example, (z_3) says that the integral

$$\int_{F_\infty} f_\infty(x_\infty) \prod_{v|\infty} |x_v|_v^{\sigma-1} \mathrm{d}x$$

where $F_\infty = \prod_{v|\infty} F_v$, converges absolutely for $\sigma > 1$. In fact, it holds for $\sigma > 0$ and all $f_\infty \in \mathcal{S}(F_\infty)$. Hence the same condition holds for $\widehat{f}$.

In the case where $f_\infty = \prod f_v^0$ with

$$f_v^0(x) = e^{-\pi x^2} \quad \text{(real variable)}$$
$$f_v^0(x) = e^{-2\pi\|z\|^2} \quad \text{(complex variable)},$$

$Z(f, s)$ is the zeta function $\zeta_F(s)$, multiplied by the usual archimedean factors (product of Γ functions) and $|D_F^{-1/2}|$. Following Tate [6], we write

$$Z(f, s) = \int_{I_F^+} f(x)|x|^s \mathrm{d}^\times x + \int_{I_F^+} \widehat{f}(x)|x|^{1-s} \mathrm{d}^\times x + \kappa \frac{\widehat{f}(0)}{s-1} - \kappa \frac{f(0)}{s} \tag{4.3}$$

following the usual notations [6, Théorème 4.3.2],

$$\kappa = \frac{2^{r_1}(2\pi)^{r_2} hR}{\sqrt{|D_F|}w}$$

is the residue of $\zeta_F(s)$ at $s = 1$. In particular, D_F is the absolute discriminant of F, and $d = r_1 + 2r_2$, where r_1 is the number of real places and r_2 is the number of complex places. Then the two integrals in (4.3) converges absolutely for all $s \in \mathbb{C}$.

Lemma 4.1. Let s be a zero of $\zeta_F(s)$ with $\Re(s) > 0$. Then $Z(f, s)$ vanishes for all $f_\infty \in \mathcal{S}(F_\infty)$.

Indeed, we can write $Z(f,s)$ for $\Re(s) > 1$ as

$$Z(f,s) = |D_F|^{-1/2} Z(f_\infty, s)\zeta_F(s).$$

Since $Z(f,s)$, $\zeta_F(s)$, and $Z(f_\infty, s)$ are holomorphic for $s \neq 1$ and $\Re(s) > 0$, the Lemma follows.

For every finite place v, $\widehat{f_v^0}$ is equal to $|\mathfrak{d}_v|^{-1/2}\mathbb{1}_{\mathfrak{d}_v^{-1}}$. Here $\mathfrak{d}_v \subset F_v$ is the different, $\mathfrak{d}_v^{-1}$ is inverse, $\mathbb{1}_{\mathfrak{d}_v^{-1}}$ is the characteristic function, and $|\mathfrak{d}_v|$ is the ideal norm (a positive power of q_v). Recall that

$$\prod_{v \text{ finite}} |\mathfrak{d}_v| = |D_F|.$$

Consider the first integral of (4.3):

$$\int_{I_F^+} f(x)|x|^s \mathrm{d}^\times x. \tag{4.4}$$

If $f(x) \neq 0$ for $x = (x_\infty, x_f)$, the decomposition $f_f = \prod_{v \text{ finite}} f_v$ shows $|x_f| \leq 1$; since $|x_\infty x_f| \geq 1$,

$$|x_\infty| = \prod_{v|\infty} |x_v| \geq 1. \tag{4.5}$$

For the second integral, we have $|x_v| \leq |\mathfrak{d}_v|$ if $x_v \in \mathfrak{d}_v^{-1}$, so $|x_f| \leq \prod_v |\mathfrak{d}_v| = |D_F|$ and

$$|x_\infty| \geq |D_F|^{-1}. \tag{4.6}$$

Lemma 4.2. Suppose that there exists a Fourier pair $(f, \widehat{f})$ on $F_\infty = \mathbb{R}^d$ such that $f(x_\infty) \geq 0$ if $|x_\infty| \geq 1$, f is strictly positive on the neighborhood of 1 in the set $|x_\infty| \geq 1$, $\widehat{f}(y_\infty) \geq 0$ if $|y_\infty| \geq D_F^{-1}$ and $f(0) = \widehat{f}(0) = 0$. Then $\zeta_F(s) \neq 0$ for all s in the interval $(0, 1)$.

(4.3) In fact (4.3) allows us to focus on its integrand: $|x|^s$ is strictly positive on the domain of integration, and the integral (4.4) is strictly positive by the assumptions on f. Hence $Z(f,s) > 0$ and $\zeta_F(s) \neq 0$ follows from Lemma 4.1.

Let $x = (x_v) \in F_\infty$. The Euclidean norm compatible with Tate's Fourier transform is

$$\|x\|^2 = \sum_{v \text{ real}} |x_v|^2 + 2 \sum_{v \text{ complex}} \|x_v\|^2.$$

Since

$$|x|^2 = \prod_{v \text{ real}} |x_v|^2 \prod_{v \text{ complex}} \|x_v\|^4,$$

arithmetic-geometric mean inequality gives

$$|x|^{2/d} \leq \frac{1}{d}\|x\|^2$$

For $r = \|x\|$, $\rho = \|y\|$ ($y \in F_\infty$) we see that

$$|x| \geq 1 \Rightarrow r \geq \sqrt{d}$$
$$|y| \geq |D_F|^{-1} \Rightarrow \rho \geq |D_F|^{-1/d}\sqrt{d}$$

Proposition 4.3. Suppose that there exists a number field of degree d and discriminant D such that ζ_F has a zero in $(0,1)$. Then

$$\mathcal{B}_d \geq d|D|^{-1/d}.$$

Conversely, ζ_F has no zero if

$$d|D|^{-1/d} > \mathcal{B}_d.$$

The proof is now obvious. Suppose $d|D|^{-1/d} > \mathcal{B}_d$. As in §3, we can find radial f and $\widehat{f}$ that are nonnegative for $r \geq \sqrt{d}$ and $\rho \geq |D|^{-1/d}\sqrt{d}$. We can assume that f is strictly positive for x with $\sqrt{d} \leq \|x\| \leq \sqrt{d} + \varepsilon$. Then the assumptions for Lemma 4.2 are satisfied since $\|1\| = \sqrt{d}$.

It is difficult to find a field F satisfying the hypothesis of Proposition 4.3. However, the decomposition of $\zeta_F(s)$ in terms of Artin L-functions of Galois extensions E over F allowed Armitage to exhibit such a zero (which is $s = 1/2$, as predicted by Riemann's hypothesis). More precisely, Armitage considered an explicit extension F over $E = \mathbb{Q}(\sqrt{3(1+i)})$ of degree 12 constructed by Serre [5], which is of degree 48 over $\mathbb{Q}$ and satisfies $\zeta_F\left(\frac{1}{2}\right) = 0$ [1, §4].

As a consequence, we have a weaker version of Theorem 3.1 from number theory.

Proposition 4.4. For d multiple of 48, $\mathcal{B}_d$ is strictly positive.

For $d = 48$, this follows from the existence of F. Assume that $d = 48c$. There exists a cyclotomic extension L over $\mathbb{Q}$ linearly disjoint with F. Then LF has degree d over $\mathbb{Q}$, and ζ_F divides ζ_{LF} since LF/F is abelian, and ζ_{LF} factorizes as a product of Dirichlet L-functions over F. Hence the result follows.

You may wonder if Proposition 4.4 provides any restriction on the discriminant of a number field where ζ_F has a real zero. In this case, we have

$$|D|^{1/d} \geq \frac{d}{\mathcal{B}_d}. \tag{4.7}$$

By Theorem 3.1,

$$\frac{d}{\mathcal{B}_d} < 2\pi e = 17.079\cdots.$$

Odlyzyko [3] proved a general unconditional bound

$$|D|^{1/d} \geq 22.2(1 + o(d))$$

for $d \to \infty$. As result we get (4.7), at least for large enough d.

Hence Proposition 4.4 does not give any interesting improvement of the lower bound of $\mathcal{B}_d$. However, it is striking to note that, at least for some degrees, number theory provides a linear growth in d given by Theorem 3.1. Let p be a prime number By theorems of Golod-Shafarevič and Brumer, there exists a tower of number fields

$$E_p^1 \subset E_p^2 \subset \cdots \subset E_p^n \subset \cdots$$

where E_p^1, that has degree $p(p-1)$ over $\mathbb{Q}$, is a degree p extension of $\mathbb{Q}(\zeta_p)$, and E_p^{n+1}/E_p^n is unramified are unramified extensions of degree p. See [4, Cor 7]; we adjoint ζ_p by two successive abelian extensions of $\mathbb{Q}$ to obtain E_p^1.

Consider the series of extensions $F_i = FE_p^i$ of F_i, where F_{i+1}/F_i is abelian of degree 1 or p. We can extract a minimal, strictly increasing subsequence

$$F_0 = FE_p^{n_0} \subset F_1 \subset \cdots \subset F_m = FE_p^{n_m}$$

where each extension is abelian of degree p. Since the extensions are all relatively unramified, a classical formula for absolute discriminants gives

$$D_{F_m} = D_{F_0}^{p^m} =: D^{p^m}. \tag{4.8}$$

The successive extensions of F are abelian, so ζ_F divides ζ_{F_m} for all m. Then Proposition 4.3 shows that for $d = d_0 p^m$, $d_0 = [F_0 : \mathbb{Q}]$:

$$\mathcal{B}_d \geq Cd, \quad C = |D|^{-1/d_0}. \tag{4.9}$$

For such degress, (3.11) and (4.9) shows that the growth of $\mathcal{B}_d$ - so is $B_d \geq \frac{1}{2}\mathcal{B}_d$, is linear in d. If p does not divide D_F, F and $\mathbb{Q}(\zeta_p)$ are linearly disjoint and we can choose E_p^1 to be linearly disjoint with F. Then $F_0 = FE_p^1$ and the inequality (4.8) is valid for $d = 48(p-1)p^n$, $n \geq 1$. Of course, the $(p-1)$ term is not necessary if one assumes Artin' conjecture or Dedekind's divisibility conjecture. (Dedekind's conjecture claims that $\zeta_F(s)$ is divisible by $\zeta_E(s)$ for all extensions E/F. Then you can choose E_p^1, possibly non-Galois, to be degree p over $\mathbb{Q}$. Then the Artin's conjecture on the holomorphicity of non-abelian L-functions implies Dedekind's conjecture.)

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