L-indistinguishable representations and trace formula for SL(2)

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Introduction

The results announced in this paper rest to a large extent on research done by R. P. Langlands and the author in Bonn in May and June 1971. We shall only sketch the proofs, and state some conjectures.

Let F be a global field and $\mathbb{A}$ be the ring of adeles of F. Put $G = \mathrm{GL}(2)$ and $S = \mathrm{SL}(2)$. Let us denote by L_0^2 the space of cusp forms on $[S] = S(\mathbb{A})/S(F)$ and by ρ the natural representation of $S(\mathbb{A})$ on L_0^2. If g is an element of $G(\mathbb{A})$ one can define a representation ρ^g of $S(\mathbb{A})$ on L_0^2 by

$$\rho^g(s) = \rho(gsg^{-1}), \quad s \in S(\mathbb{A}).$$

The problem to be solved here answers a question asked by R. P. Langlands: "Is the representation ρ^g equivalent to ρ, and if not how do they differ?" We shall show that in general they are not equivalent and give some explicit expression of their "difference". The result and the method ot prove ti were guessed by Langlands. It is a direct application of Selberg's Trace Formula for SL(2), and even merely a way of writing it. The complete knowledge of this Trace Formula leads to some conjectures, namely the description of the spectrum of ρ in terms of the spectrum of the natural representation of $G(\mathbb{A})$ in the space of cusp forms on $G(\mathbb{A})/G(F)$ studied by Jacquet and Langlands in [4].

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1 Arithmetic equivalence

The group $S(\mathbb{A})$ is an invariant subgroup in $G(\mathbb{A})$, so that if g is an element of $G(\mathbb{A})$ and U any unitary representation of $S(\mathbb{A})$, we can define a unitary representation U^g of $S(\mathbb{A})$ by

$$U^g(s) = U(gsg^{-1}), \quad s \in S(\mathbb{A}).$$

The two representations U and U^g will be called *arithmetically equivalent*. If U is irreducible then U and U^g will also sometimes be called *L-indistinguishable*; in fact from some arithmetical points of view they seem indistinguishable despite the fact they are not in general equivalent in the ordinary sense. One must remark that if U occurs in the restriction V to $S(\mathbb{A})$ of some representation $\widetilde{V}$ of $G(\mathbb{A})$, then U^g will occur in V with the same multiplicity for any $g \in G(\mathbb{A})$.

There is an obvious local analogue of this global arithmetical equivalence. For instance a member of the discrete series of representations of $\mathrm{SL}(2,\mathbb{R})$ and its complex conjugate are arithmetically equivalent in the local sense. It is well known that the representation ρ of $S(\mathbb{A})$ in L_0^2, splits into a direct sum of irreducible representations with finite multiplicities. Let us denote by $\widehat{S(\mathbb{A})}$, the set of equivalence classes of irreducible unitary representations of $S(\mathbb{A})$, and by $m(U)$ the multiplicity of $U \in \widehat{S(\mathbb{A})}$ in ρ, so that

$$\rho = \sum_{U \in \widehat{S(\mathbb{A})}} m(U)U.$$

In the same way, given any $g \in G(\mathbb{A})$, we have

$$\rho^g = \sum_{U \in \widehat{S(\mathbb{A})}} m(U)U^g = \sum_{U \in \widehat{S(\mathbb{A})}} m(U^{g^{-1}})U.$$

We see that the comparison of ρ and ρ^g amounts to comparing $m(U)$ and $m(U^g)$. Our problem can be restated ni the folowing way: do U and U^g occur with the same multiplicity (may be zero) in ρ? We are thus led to study the continuous functions

$$g \mapsto m(U^g)$$

from $G(\mathbb{A})$ to the set $\mathbb{N}$ of natural numbers. If this function is constant (resp. non-constant) the representation U will be called *stable* (resp. *unstable*).

Denoting by Z with the center of G, we have

Lemma 1.1. If g belongs to $Z(\mathbb{A})S(\mathbb{A})G(F)$, then ρ^g is equivalent to ρ.

Proof. If g lies in $S(\mathbb{A})$ or in $Z(\mathbb{A})$ the statement is obvious. If g lies in $G(F)$ the conjugation by g defines an automorphism of the space $S(\mathbb{A})/S(F)$, and so determines an intertwining operator between ρ and ρ^g, hence the lemma. □

As a corollary we see that the function $g \mapsto m(U^g)$ is constant on the invariant subgroup $S(\mathbb{A})Z(\mathbb{A})G(F)$ of $G(\mathbb{A})$. If we denote by $\mathbb{I}$ the group of ideles of F, by $\mathbb{I}^2$ the group of square ideles and by $F^\times$ the multiplicative group of F, then $G(\mathbb{A})/Z(\mathbb{A})S(\mathbb{A})G(F)$ is isomorphic via the determinant mapping to the compact abelian group $\mathbb{I}/\mathbb{I}^2F^\times$. Now the function $g \mapsto m(U^g)$ induces a discrete valued continuous function on a compact group and hence assumes only finitely many values.

Now let ε be an element of $\mathcal{E}$ the group of characters of $\mathbb{I}/\mathbb{I}^2F^\times$. We can define

$$m_\varepsilon(U) = \int_{G(\mathbb{A})/S(\mathbb{A})Z(\mathbb{A})G(F)} \varepsilon(\det g)m(U^g)\mathrm{d}g$$

where $\mathrm{d}g$ is the Haar measure with total mass one.

We have the following

Lemma 1.2. 1. A representation $U \in \widehat{S(\mathbb{A})}$ is stable if and only if $m_\varepsilon(U) = 0$ for any nontrivial $\varepsilon \in \mathcal{E}$.

2. A representation $U \in \widehat{S(\mathbb{A})}$ is unstable if and only if $m_\varepsilon(U) \neq 0$ for some nontrivial $\varepsilon \in \mathcal{E}$.

3. $m(U) = \sum_{\varepsilon\in\mathcal{E}} m_\varepsilon(U)$, the series benig only a sum over a finite subgroup of $\mathcal{E}$ depending on U.

The study of the numbers $m_\varepsilon(U)$ will now be made using Selberg's trace formula.

2 The trace formula

We shall denote by $\mathcal{S}(S(\mathbb{A}))$ the space of Schwartz-Bruhat functions with compact support on $S(\mathbb{A})$. It follows from [1] that for any $f \in \mathcal{S}(S(\mathbb{A}))$, the operator $\rho(f)$ is of trace class and its trace is given by Selberg's Trace Formula. Given $f \in \mathcal{S}(S(\mathbb{A}))$ and $g \in G(\mathbb{A})$ we define a function f^g on $S(\mathbb{A})$ by

$$f^g(s) = f(gsg^{-1});$$

we have $\mathrm{Tr}\rho^g(f) = \mathrm{Tr}\rho(f^{g^{-1}})$; moreover the function

$$g \mapsto \mathrm{Tr}\rho(f^g)$$

is continuous and constant on the cosets modulo $S(\mathbb{A})Z(\mathbb{A})G(F)$, so that one can define

$$I_\varepsilon(f) = \int_{G(\mathbb{A})/S(\mathbb{A})Z(\mathbb{A})G(F)} \varepsilon(\det g)\mathrm{Tr}\rho(f^g)\mathrm{d}g.$$

This integral is merely a finite sum, and is zero except for finitely many values of $\varepsilon \in \mathcal{E}$. On the other hand we have

$$\mathrm{Tr}\rho(f) = \sum_{U \in \widehat{S(\mathbb{A})}} m(U)\mathrm{Tr}U(f),$$

the series being absolutely convergent. It is then clear that

$$I_\varepsilon(f) = \sum_{U \in \widehat{S(\mathbb{A})}} m_\varepsilon(U)\mathrm{Tr}U(f)$$

and that

$$\mathrm{Tr}\rho(f) = \sum_{\varepsilon \in \mathcal{E}} I_\varepsilon(f).$$

The values of the factors $m_\varepsilon(U)$ will be deduced from the explicit knowledge of the integrals $I_\varepsilon(f)$ for sufficiently many functions f, in application of the principle that a representation is determined by its trace. Before we explain the computation of the integrals $I_\varepsilon(f)$, let us state the result.

3 The formula for $I_\varepsilon(f)$

Throughout this paragraph we fix a nontrivial character $\varepsilon \in \mathcal{E}$. Global class field theory associates to ε a separable quadratic extension L of F. We shall denote by E the kernel of the norm mapping $\mathrm{N}_{L/F}$. The algebraic group E is an F-anisotropic torus of dimension one; we shall denote by $\Theta(\varepsilon)$ the dual group of the compact group $E(\mathbb{A})/E(F)$. If ψ is a nontrivial character of $\mathbb{A}/F$, then using the Weil representation one can define for any character θ of $E(\mathbb{A})$, a representation $U(\psi,\theta)$ of $S(\mathbb{A})$ (cf. Shalika and Tanaka [5]). The equivalence class of $U(\psi,\theta)^g$ depends only on $d = \det(g)$ and we shall denote it by $U(\psi,\theta,d)$; moreover two such equivalence classes for d and d' in $\mathbb{I}$ coincide if and only if $d'd^{-1} \in \mathrm{N}_{L/F}(\mathbb{I}_L)$. We recall that $\mathrm{N}_{L/F}(\mathbb{I}_L)$ is an open subgroup of $\mathbb{I}$. Let us now define

$$U_\theta^+ = \sum_{\substack{d \in \mathbb{I}/\mathrm{N}_{L/F}(\mathbb{I}_L) \\ \varepsilon(d)=1}} U(\psi,\theta,d)$$

and

$$U_\theta^- = \sum_{\substack{d\in\mathbb{I}/\mathrm{N}_{L/F}(\mathbb{I}_L)\\ \varepsilon(d)=-1}} U(\psi,\theta,d).$$

The representations U_θ^+ and U_θ^-, are independent of the choice of ψ; moreover their intertwining number is zero. We can now state the

Theorem 3.1. Let ε be a nontrivial character of $\mathbb{I}/\mathbb{I}^2F^\times$, and let $f \in \mathcal{S}(S(\mathbb{A}))$, then:

(i) The operators $U_\theta^+(f)$ and $U_\theta^-(f)$ are of trace class.

(ii) The function $\theta \mapsto [\mathrm{Tr}U_\theta^+(f) - \mathrm{Tr}U_\theta^-(f)]$ is a Schwartz–Bruhat function on the dual $\widehat{E(\mathbb{A})}$ of $E(\mathbb{A})$.

(iii)

$$I_\varepsilon(f) = \frac{1}{4}\sum_{\substack{\theta\in\Theta(\varepsilon)\\ \theta\neq 1}} [\mathrm{Tr}U_\theta^+(f) - \mathrm{Tr}U_\theta^-(f)].$$

Proof. The proof of assertion (iii) will be outlined in Section 4; let us only observe now that assertion (ii) implies the absolute convergence of the series. To prove (i) and (ii) we may asume that f is adecomposable function $f = \otimes_v f_v$ where for almost all v (the places of F) $f_v = \chi_v$ the characteristic function of the standard maximal compact subgroup K_v of $S_v = \mathrm{SL}(2, F_v)$. Now if v is a finite place of F and if U_v is an irreducible unitary representation of S, then $U_v(\chi_v) = 0$ unless U_v contains the trivial representation of K_v. The representation $U(\psi,\theta,d)$ is a tensor product of representations $U(\psi_v,\theta_v,d_v)$ and if v is finite then $U(\psi_v,\theta_v,d_v)(\chi_v) = 0$ if θ_v is ramified (i.e. if θ_v is nontrivial on the maximal compact subgroup of E_v). If v is finite and if θ_v is unramified then $\mathrm{Tr}U(\psi_v,\theta_v,d_v)(\chi_v)$ is independent of θ_v and can only assume the values +1 or 0 if the local Haar measure is such that $\mathrm{vol}(K_v) = 1$, depending on the choice of ψ_v, d_v, and the structure of L_v. In any case $\theta_v \mapsto \mathrm{Tr}U(\psi_v,\theta_v,d_v)$ is a Schwartz–Bruhat function on $\widehat{E_v}$. We deduce from these facts that $\theta \mapsto \mathrm{Tr}U(\psi,\theta,d)$ is a Schwartz–Bruhat function on $\widehat{E(\mathbb{A})}$. But one can show that outside a finite subgroup of $\mathbb{I}/\mathrm{N}_{L/F}(\mathbb{I}_L)$, independent of θ then $U(\psi,\theta,d)(f) = 0$, and hence the assertions (i) and (ii) are proved. □

Corollary 3.2. Let $U \in \widehat{S(\mathbb{A})}$ then $m_\varepsilon(U) = 0$ or $|m_\varepsilon(U)| \geq 1/4$.

Using a slightly generalized form of the Lemma 16.1.1, p. 195 in [4] one see that if $m_\varepsilon(U) \neq 0$ then U is equivalent to an irreducible part of some $U(\psi,\theta,d)$, and the corollary is proved. More precise information can be obtained from the knowledge of the decomposition of the representations $U(\psi,\theta,d)$ and of the intertwining operators between $U(\psi,\theta,d)$ and $U(\psi,\theta',d')$, so that one obtains a complete classification of the unstable representations and the exact value of $m_\varepsilon(U)$. Let us simply say here that the local representations $U(\psi_v,\theta_v,d_v)$ are irreducible if $\theta_v = 1$ or if $\theta_v^2 \neq 1$ and that the intertwining number between $U(\psi_v,\theta_v,d_v)$ and $U(\psi_v,\theta'_v,d'_v)$ is zero unless $d_v = d'_v$ and $\theta_v = \theta'_v$ or $\theta_v^{-1} = \theta'_v$ in which case they are equivalent. If $\theta_v \neq 1$ and $\theta_v^2 = 1$ then $U(\psi_v,\theta_v,d_v)$ splits into two inequivalent irreducible parts. One must remark that if $U \in \widehat{S(\mathbb{A})}$, and if $m_\varepsilon(U) \neq 0$ for some $\varepsilon \in \mathcal{E}$ then there exists at least one $g \in G(\mathbb{A})$ such that $m(U^g) \neq 0$ and we thus get a new proof (but in a less explicit way) of the result of Shalika and Tanaka [5] according ot which certain representations of $S(\mathbb{A})$ attached ot global characters of a nonsplit F-torus occur in ρ. (See also [4] p. 396.)

4 Computation of $I_\varepsilon(f)$

We shall denote by r the natural representation of $S(\mathbb{A})$ on the space L^2 of square integrable functions on $S(\mathbb{A})/S(F)$; the representation ρ is the restriction of r to the invariant subspace L^2_0. The orthogonal complement of L^2_0 in L^2 is the direct sum of the one dimensional space of constant functions, and of a space L^2_1 on which r induces a representation equivalent to a continuous integral of representations of the global principal series (i.e. representations associated to unitary Hecke characters of a F-split torus). This spectral decomposition is constructed by the use of Eisenstein series (cf. [3], see also [1], [2] and [4]) in an explicit way, and one finds that if δ denotes the unit representation of $S(\mathbb{A})$, then

$$\mathrm{Tr}\rho(f) + \mathrm{Tr}\delta(f) = \int_{[S]} [K_f(s,s) - H_f(s,s)]\,\mathrm{d}s$$

where

$$K_f(x,y) = \sum_{\gamma\in S(F)} f(x\gamma y^{-1})$$

is the kernel of $r(f)$ in L^2 and where $H_f(x,y)$ is the kernel of the operator $r(f)$ restricted to L^2_1. Let us denote by S_e the set of elliptic elements in $S(F)$ (i.e. the set of elements whose eigenvalues do not lie in F) and by S_p its complementary set, the set of parabolic elements. The above integral can be split into two convergent parts. In fact it is easy to show that the integral

$$J(f) = \int_{[S]} \sum_{\gamma\in S_e} f(s\gamma s^{-1})\,\mathrm{d}s$$

is absolutely convergent. The other part is

$$C(f) = \int_{[S]} \sum_{\gamma \in S_p} f(s\gamma s^{-1}) - H_f(s,s)\,\mathrm{d}s$$

and is called the complementary term (c.f. [1]). We now introduce the integrals

$$J_\varepsilon(f) = \int_{G(\mathbb{A})/Z(\mathbb{A})E(F)} \varepsilon(\det g) \sum_{\gamma \in S_e} f(gs\gamma g^{-1})\,\mathrm{d}g.$$

The invariant measures are chosen in such a way that

$$J(f) = \sum_{\varepsilon \in \mathcal{E}} J_\varepsilon(f).$$

Now fix some non-trivial $\varepsilon \in \mathcal{E}$, and taking over the notations of §3 choose some embedding of $L^\times$ in $G(F)$; it is not difficult to prove that

$$J_\varepsilon(f) = \frac{1}{4}\mathrm{vol}([E]) \sum_{g \in G(\mathbb{A})/S(\mathbb{A})\mathbb{I}_L} \varepsilon(\det g) \int_{S(\mathbb{A})/E(\mathbb{A})} \sum_{\substack{\gamma \in E(F) \\ \gamma \neq \pm 1}} f(gs\gamma s^{-1}g^{-1})\,\mathrm{d}s.$$

We now introduce local factors:

$$F^\varepsilon_{f_v}(e_v) = \sum_{g \in G_v/S_v L_v^\times} \varepsilon_v(\det g) \int_{S_v/E_v} a_v(e_v) f_v(gse_v s^{-1} g^{-1})\,\mathrm{d}s$$

where

$$a_v(e_v) = b_v(e_v)|\mathrm{N}_{L/F}(e_v - e_v^{-1})|_v^{1/2}$$

and

$$b_v\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \varepsilon_v(y).$$

The function $e_v \mapsto F_{f_v}(e_v)$ is a priori defined only when $e_v \neq \pm 1$ but it can be continued in a Schwartz–Bruhat function on E_v. This can be checked directly or by computing its Fourier transform, namely

$$\widehat{F^\varepsilon_{f_v}}(\theta_v) = \lambda(L_v/F_v, \psi_v) \sum_{d \in F_v/\mathrm{N}_{L/F}(L_v^\times)} \varepsilon_v(d)\mathrm{Tr}U(\psi_v, \theta_v, d)(f_v)$$

(c.f. [2]) where $\lambda(L_v/F_v, \psi_v)$ is a factor of absolute value one defined in [4] p. 6. Using the assertions (i) and (ii) of the theorem in §3, we have then proved that

$$\widehat{F^\varepsilon_f}(\theta) = \prod_v \widehat{F^\varepsilon_{f_v}}(\theta_v) = \mathrm{Tr}U^+_\theta(f) - \mathrm{Tr}U^-_\theta(f)$$

is a Schwartz-Bruhat function. (We recall that the factors $\lambda(L_v/F_v, \psi_v)$ have a product equal to 1.) Now applying Poisson's summation formula we obtain

$$J_\varepsilon(f) = \frac{1}{4}\sum_{\theta\in\Theta(\varepsilon)} [\mathrm{Tr} U_\theta^+(f) - \mathrm{Tr} U_\theta^-(f)] - (F_f^\varepsilon(1) + F_f^\varepsilon(-1))\frac{\mathrm{vol}([E])}{4}$$

if the base field is of characteristic zero or $p \neq 2$; if the characteristic is 2 one should read $F_f^\varepsilon(1)$ instead of $F_f^\varepsilon(1) + F_f^\varepsilon(-1)$.

Let $K = \prod_v K_v$ be the maximal compact subgroup of $S(\mathbb{A})$. If $s = k\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}\begin{pmatrix} 1 & u \\ & 1\end{pmatrix}$ is an Iwasawa decomposition of $s \in S(\mathbb{A})$, a Haar measure on $S(\mathbb{A})$ will be given by $\mathrm{d}s = |t|^2 \mathrm{d}k \mathrm{d}^\times t \mathrm{d}u$. The measures $\mathrm{d}k$ and $\mathrm{d}^\times t$ are left arbitrary but $\mathrm{d}u$ wil be the Tamagawa measure on $\mathbb{A}$ (i.e. $\mathrm{vol}(\mathbb{A}/F) = 1$). If we now choose the measure on $E(\mathbb{A})$ in such a way that $\mathrm{vol}([E]) = 2$ we find

$$F_f^\varepsilon(\pm 1) = \prod_v F_{f_v}^\varepsilon(\pm 1) = \lim_{\sigma\to 1}\int_{\mathbb{I}} f^K\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1\end{pmatrix}\varepsilon(t)|t|^\sigma\, \mathrm{d}^\times t$$

where

$$f^K(s) = \int_K f(ksk^{-1})\,\mathrm{d}k.$$

We now have to study the complementary term $C(f)$. One can exhibit expressions $C_\varepsilon(f)$ such that

$$C(f) = \sum_{\varepsilon\in\mathcal{E}} C_\varepsilon(f)$$

with

$$C_\varepsilon(f^g) = \varepsilon(\det g) C_\varepsilon(f) \quad \text{for all } g \in G(\mathbb{A}).$$

Let (φ_i) be a sequence of Schwartz–Bruhat functions on $S(\mathbb{A})$ converging to the constant function 1 in L^2 defined like in [1] p. 235 and put

$$C'(f, \varphi_i) = \int_{[S]} \varphi_i(s) \sum_{\gamma\in S_p} f(s\gamma s^{-1})\,\mathrm{d}s.$$

The evaluation of such integrals leads to the computation of the residue at $z = 1$ of the analytic continuation of the functions

$$Z(f, z; a) = \int_{\mathbb{A}}\int_{\mathbb{I}/F^\times}\sum_{\eta\in F^\times} f^K\begin{pmatrix} a & (a-a^{-1})u + t^2 u \\ 0 & a^{-1}\end{pmatrix}\frac{|t|^{z+1}}{R(u)^{z+1}}\mathrm{d}^\times t\mathrm{d}u$$

if $\Re(z) > 1$ where $R(u)$ is defined in [1] p. 244. One sees immediately that

$$Z(f, z; a) = \sum_{\varepsilon\in\mathcal{E}} \widetilde{Z}(f, z, \varepsilon; a)$$

where

$$\widetilde{Z}(f,z,\varepsilon;a) = \frac{1}{2}\int_{\mathbb{A}}\int_{\mathbb{I}} f^K \begin{pmatrix} a & (a-a^{-1})u+t \\ 0 & a^{-1} \end{pmatrix} \frac{\varepsilon(t)|t|^{\frac{z+1}{2}}}{R(u)^{z+1}}\, \mathrm{d}^\times t \mathrm{d}u$$

if $\Re z > 1$. These integrals are introduced in [1] p. 243 with similar notation and are studied in detail when $\varepsilon = 1$, the trivial character. When $\varepsilon \neq 1$ one can show that if $a \neq a^{-1}$ then $\widetilde{Z}(f,z,\varepsilon;a)$ is holomorphic at $z = 1$ and hence gives no contribution. If $\varepsilon \neq 1$ and $a = a^{-1}$ then $\widetilde{Z}(f,z,\varepsilon;a)$ has a simple pole at $z = 1$ and gives a contribution to $C(f', \varphi_i)$ which with our choice of the Haar measures is

$$\frac{1}{2}\lim_{z\to 1}\int_{\mathbb{I}} f^K \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} |t|^{\frac{z+1}{2}}\varepsilon(t)\, \mathrm{d}^\times t$$

and is in $C_\varepsilon(f)$. (This contribution is Independent of φ_i because we assume that $\int_{[S]} \varphi_i(s)\mathrm{d}s = \mathrm{vol}([S])$ for all i.)

The computation of the remaining part

$$C''(f,\varphi_i) = -\int_{[S]} \varphi_i(s) H_f(s,s)\, \mathrm{d}s$$

can be done exactly as in [1]. In the result appears, among other terms, the expression $-\frac{1}{4}\mathrm{Tr}\widehat{f}(\varepsilon)M(\varepsilon)$ where $\widehat{f}(\varepsilon)$ is the convolution operator by f in the space of the representation of the global principal series for $S(\mathbb{A})$ relative to the character ε; and $M(\varepsilon)$ is an intertwining operator of that representation. One can show that this term is in $C_\varepsilon(f)$ and that if $\varepsilon \neq 1$ then

$$C_\varepsilon(f) = \frac{1}{2}\lim_{\sigma\to 1}\int_{\mathbb{I}} \sum_{\substack{a^2=1 \\ a\in F^\times}} f^K \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \varepsilon(t)|t|^\sigma\, \mathrm{d}^\times t - \frac{1}{4}\mathrm{Tr}(\widehat{f}(\varepsilon)M(\varepsilon)).$$

Assertion (iii) of the theorem in §3 then is the consequence of the relation

$$\mathrm{Tr}\widehat{f}(\varepsilon)M(\varepsilon) = \mathrm{Tr}U^+_{\theta_0}(f) - \mathrm{Tr}U^-_{\theta_0}(f)$$

where θ_0 is the unit element of $\Theta(\varepsilon)$.

5 Conjectures

We do not give here the expression for the "stable" term of the trace formula. Let us simply say it is very close to the complete trace formula for $G(\mathbb{A})$. I am led to formulate the following conjectures:

(i) A representation $U \in \widehat{S(\mathbb{A})}$ occurs in ρ^g for some $g \in G(\mathbb{A})$, if and only if U occurs in the restriction ot $S(\mathbb{A})$ of some representation $\widetilde{U}$ of $G(\mathbb{A})$ occuring in the space of cusp forms for $G(\mathbb{A})$ relative to some character of the centre.

(ii) The multiplicity one theorem is true for the representation of $S(\mathbb{A})G(F)$ on $L_0^2(S(\mathbb{A})G(F)/G(F)) = L_0^2(S(\mathbb{A})/S(F))$.

(iii) If $U \in \widehat{S(\mathbb{A})}$ is unstable $m(U) = 1$ if and only if $m_\varepsilon(U) \geq 1$ for all $\varepsilon \in \mathcal{E}$ and $m(U) = 0$ otherwise.

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