Bruhat–Tits buildings, Moy–Prasad filtration, and Yu's construction

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Abstract

This is an expository note on the theory of Bruhat–Tits buildings, and appplications to the Moy–Prasad filtration and Yu's construction, with no proofs. The main references of this note are Rabinoff's senior thesis [17] and Fintzen's CDM & IHES lecture notes [5, 4].

1 Introduction

One of the fundamental goal of the representation theory is to classify all the irreducible representations of a given group. In this seminar, ~~we are~~ I am interested in the representations of p-adic groups, such as $\mathrm{SL}_n(\mathbb{Q}_p)$ or $\mathrm{Sp}_{2n}(\mathbb{Q}_p)$. As in the finite group case, usual way to construct representations is by (parabolic, compact, ...) induction. From this viewpoint, it is natural to ask what are the "building blocks" of irreducible representations that might exhaust all the irreducible representations by inducting them. The answer will be *supercuspidal representations*.

Definition 1.1. An irreducible admissible representation (π, V) of $G(F)$ is called *supercuspidal* if every Jacquet module $V_N := V/\langle \pi(n)v - v : n \in N, v \in V\rangle$ for parabolic $P = MN \subset G$ is zero. Equivalently, their matix coefficients are compactly supported modulo the center.

Supercuspidal representations are building blocks in the following sense:

Theorem 1.2 (Bernstein [1])**.** Let (π, V) be an irreducible smooth representation of G. Then there exists a parabolic subgroup $P \subseteq G$ with Levi subgroup M and a supercuspidal representation (σ, W) of $M(F)$ such that (π, V) is a subrepresentation of $(\mathrm{Ind}_{P(F)}^{G(F)}\sigma, \mathrm{Ind}_{P(F)}^{G(F)}W)$.

So, how can we construct supercuspidal representations? One way to do (at least, one way to construct *a* representation in an interesting way) is to use induction from a compact subgroup. For example, consider the following diagram:

$$\begin{array}{ccc} \mathrm{GL}_2(\mathbb{Q}_p) & & \\ \uparrow & & \\ \mathrm{GL}_2(\mathbb{Z}_p) & \twoheadrightarrow & \mathrm{GL}_2(\mathbb{F}_p) \end{array}$$

Let's assume that representations of finite groups are "easier" to understand.[1] Once we have an irreducible representation ρ of $\mathrm{GL}_2(\mathbb{F}_p)$ (which is a finite group), we can pullback it to $\mathrm{GL}_2(\mathbb{Z}_p)$ and then induce it to $\mathrm{GL}_2(\mathbb{Q}_p)$. Then we may prey that the induced representation is irreducible, or even supercuspidal. This works sometimes - for example, if ρ is a *Weil representation* of $\mathrm{GL}_2(\mathbb{F}_p)$, then the pullbacked & induced representation is supercuspidal (you may find details in [3, Chapter 4]). Howe [11] give a construction of a lot of supercuspidal representations of $\mathrm{GL}_n(F)$, which turned out to be exhaustive by Moy [14].

It is a folklore conjecture that all supercuspidal representations (of general reductive groups) arise via compact induction from a representation of a compact-mod-center open subgroup. Although we don't know how to prove the conjecture yet, most of the known constructions of the supercuspidal representations would be based on the idea. Hence it is important to know what compact-mod-center subgroups we need to consider and what representations of them we need to induce. The naive answer that we'll see is the following:

> For each G, there exists a contractible complete metric space $\mathcal{B}(G)$ called *Bruhat–Tits building* where $G(\mathbb{Q}_p)$ acts nicely. Each point $x \in \mathcal{B}(G)$ gives a *Moy–Prasad filtration* $\{G_{x,r}\}_{r\in\mathbb{R}_{\geq 0}}$, compact open subgroups of the stabilizer subgroup $G_x \subset G$, whose quotients are Lie groups of finite type or abelian groups. Yu's construction produces a supercuspidal representation to *Yu's datum*, which consists of (i) a finite filtration G, (ii) a vertex in $\mathcal{B}(G)$, (iii) depth-zero cuspidal representation of the smallest subgroup in the filtration, and (iv) *generic* characters of the filtration subgroups.

[1]Maybe not... but at least for small p's, you can compute them by hand. In general, you may need Deligne–Lusztig theory.

2 Bruhat–Tits building

When we study representations of Lie groups G (especially, infinite dimensional representations), it is common to consider the cohomology of the associated symmetric space $X = G(\mathbb{R})/K$ (K is a maximal compact subgroup of $G(\mathbb{R})$). For example, discrete series of $\mathrm{GL}_2(\mathbb{R})$ can be understood by line bundles on $\mathbb{H}^{\pm}$. Hence it is natural to ask if one can find an analogous space for p-adic groups to study representations, and as you expected, the answer is *Bruhat–Tits building*.

Bruhat–Tits building $\mathcal{B}(G) = \mathcal{B}(G, \mathbb{Q}_p)$ of G is a certain contractible and complete metric space obtained by glueing bunch of *apartments* (which are also called *maximal flats*), which are just Euclidean spaces, where $G(\mathbb{Q}_p)$ acts nicely.[2] Each of apartment correspond to a maximal split tori, where the bijection is given by considering a stabilizer of the apartment in $G(\mathbb{Q}_p)$. The action of $G(\mathbb{Q}_p)$ is transitive on the apartments, but not on vertices - the action is transitive on vertices *with same types*.

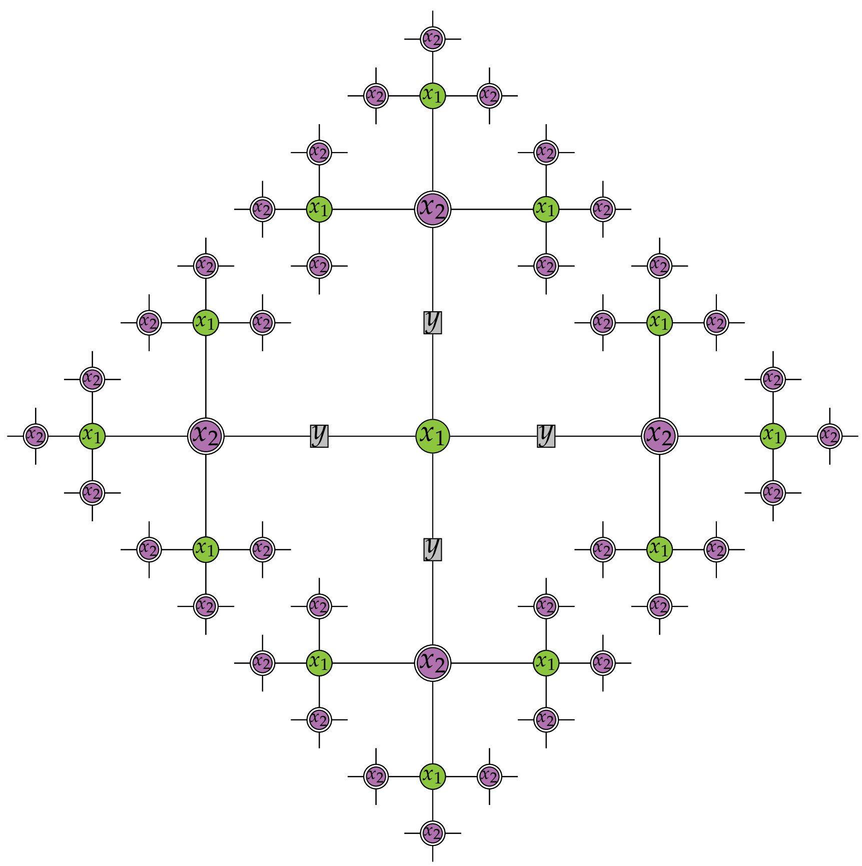

Figure 1: Bruhat–Tits building of $\mathrm{SL}_2(\mathbb{Q}_3)$

[2]In Bruhat–Tits building, people (points) are living in at least two apartments.

For example, Figure 1 shows one the most famous Bruhat–Tits building that you may find on Google, which is $\mathcal{B}(\mathrm{SL}_2(\mathbb{Q}_3))$:[3] It is an infinite 4-regular tree, and in general, $\mathcal{B}(\mathrm{SL}_2(\mathbb{Q}_p))$ is a infinite $(p+1)$-regular tree. Each vertex corresponds to an isomorphism class of *lattices* in $\mathbb{Q}_3^2$ (up to homothety by $\mathbb{Q}_3^\times$), where two of them are connected if and only if there exist representatives L, L' of each class such that $L' \subset L$ and $L/L' \simeq \mathbb{Z}/3\mathbb{Z}$. For any given L and L', there exists a basis $\{e_1, e_2\}$ of L and integers $a \geq b$ such that $\{3^a e_1, 3^b e_2\}$ form a basis of L', where $|a-b|$ is equal to the distance between the corresponding points. We have a natural action of $\mathrm{SL}_2(\mathbb{Q}_3)$ on the tree via $[L] \mapsto [gL]$ ($g \in \mathrm{SL}_2(\mathbb{Q}_3)$), and it preserves the metric. It is easy to check that the stabilizer of the point corresponds to $L_0 = \mathbb{Z}_3^2$ is $\mathrm{SL}_2(\mathbb{Z}_3)$. Note that the action is not transitive on the whole vertices, since $d([L],[gL])$ is always even. In fact, it acts transitively on the vertices with same colors in Figure 1.

Apartments of this tree are the infinite geodesics on it (which is uncountably many), where each of them corresponds to a maximal split tori, i.e. a conjugation of the diagonal torus $\begin{bmatrix} \mathbb{Q}_3^\times & \\ & \mathbb{Q}_3^\times \end{bmatrix}$. For example, one can think the apartment for the diagonal torus as a line containing all (equivalence classes of) the lattices of the form $L = \mathbb{Z}_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \mathbb{Z}_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The normalizer $N = N_G(T)$ of the diagonal torus acts on the line, where the action factors through the *affine Weyl group* $\widetilde{W} \simeq N/T(\mathbb{Z}_3) \simeq \mathbb{Z} \rtimes (\mathbb{Z}/2)$, which is generated by reflections over affine hyperplanes. See Serre's book for more details [18, Chapter II].

More generally, we define apartment of a pair (G, T) where T is a maximal torus of G simply as $\mathcal{A} = \mathcal{A}(G, T) = E^*$, the Euclidean space containing the coroots. Each apartment depends on the choice of T, and we *glue* all of them to get the building:

$$\mathcal{B}(G) := (G \times \mathcal{A})/\sim,$$

where the equivalence relation is given by $(g, x) \sim (h, y)$ iff $\exists n \in N$ such that $nx = y$ and $g^{-1}hn \in \mathrm{Stab}_G(x)$. Intuitively, we are considering all the possible reflections of $\mathcal{A}$ by $\widetilde{W}$, and glue them together through the reflections. We have a natural metric induced from that of E^*, where proving the triangle inequality is nontrivial [2].

There's more abstract way to define apartments and buildings, using affine spaces and polysimplicial complexes. Although we only focus on the Bruhat–Tits buildings coming from p-adic groups.

Here's one nice application of the theory of buildings.

[3] The other one is $\mathcal{B}(\mathrm{SL}_2(\mathbb{Q}_2))$.

Theorem 2.1 (Cartan decomposition)**.** Let $G = \mathrm{SL}_n(\mathbb{Q}_p)$ and $K = \mathrm{SL}_n(\mathbb{Z}_p)$ be a maximal compact group. Let $A \subset G$ be a maximal split torus (e.g. diagonals). Then $G = KAK$.

Proof. (This proof is stolen from Morris' note [13].) Let $h \in G(\mathbb{Q}_p)$. Let $\mathcal{A} \subset \mathcal{B}(G, \mathbb{Q}_p)$ be an apartment corresponds to A with base point $x \in \mathcal{A}$. Then there exists an apartment $\mathcal{A}'$ that contains both x and hx. Also, transitivity of the action tells us that there exists $g \in G(\mathbb{Q}_p)$ with $\mathcal{A} = g\mathcal{A}'$, which fixes the intersection $\mathcal{A} \cap \mathcal{A}'$, hence x. It means that $ghx \in \mathcal{A}$, and x & ghx have the same type, so there exists $a \in A$ such that $a(ghx) = x$. Hence g and agh both belongs to the stabilizer of x, which actually equals to K. Thus $h = g^{-1} \cdot a^{-1} \cdot (agh) \in KAK$. □

3 Moy–Prasad filtration

Recall that all supercuspidal representations are conjecturally induced from compact subgroups. Moy–Prasad filtration gives a partial answer to the question; it is a filtration of G attached to each point of $\mathcal{B}(G)$, which is used to define *depth* of representations. Especially, we can classify all the *depth-zero* supercuspidal representations, where all of them arise as inductions of *depth-zero minimal K types*.

Before we define the Moy–Prasad filtration, we introduce the notion of *parahoric subgroups*, which are groups G_x associated to each point $x \in \mathcal{B}(G)$. When $x \in \mathcal{A}(G, T)$, we define G_x and its (pro-)unipotent radical G_x^+ as follows:

$$G_x := \langle T(\mathbb{Z}_p), x_\alpha(p^{-\lfloor \alpha(x) \rfloor}) : \alpha \in \Phi \rangle \tag{1}$$

$$G_x^+ := \langle T(1 + p\mathbb{Z}_p), x_\alpha(p^{1-\lceil \alpha(x) \rceil}) : \alpha \in \Phi \rangle \tag{2}$$

Now, for an arbitary point $x \in \mathcal{B}(G)$, take $x_0 \in \mathcal{A}(G)$ and $g \in G$ with $x = g \cdot x_0$. Then we define $G_x := gG_{x_0}g^{-1}$ and $G_x^+ := gG_{x_0}^+g^{-1}$, which is independent of the choice of x_0 and g. Both of the groups only depends on the facet $\mathcal{F}$ containing x, and sometimes we denote them by $G_{\mathcal{F}}$ and $G_{\mathcal{F}}^+$. Note that the quotient G_x/G_x^+ is always a Lie group of finite type, which we denote as $\mathcal{G}_x$.

Moy–Prasad filtration is a filtration of these two groups.

Definition 3.1 (Moy–Prasad filtration)**.** Let $x \in \mathcal{A}(G, T)$. Moy–Prasad filtration of G_x and G_x^+ is given by, for each $r \in \mathbb{R}_{\geq 0}$,

$$G_{x,r} := \langle T(1 + p^{\lceil r \rceil}), x_\alpha(p^{-\lfloor \alpha(x) - r \rfloor}) : \alpha \in \Phi \rangle \subset G_x \tag{3}$$

$$G_{x,r^+} := \bigcap_{s > r} G_{x,s} \tag{4}$$

We can generalize the definition to any $x \in \mathcal{B}(G)$ similarly as before. It is easy to check that for any x, $G_{x,0} = G_x$, $G_{x,0^+} = G_x^+$, and $\{G_{x,r}\}_{r \geq 0}$ form a decreasing filtration and a basis of open compact neighborhoods of the identity in G. The parameter r of the filtration is additive in the following sense: we have $[G_{x,r}, G_{x,r'}] \subseteq G_{x,r+r'}$. Also, we have analogous filtrations $\mathfrak{g}_{x,r^+} \subset \mathfrak{g}_{x,r} \subset \mathfrak{g}_x$ for the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$.

For example, consider $G = \mathrm{SL}_2(\mathbb{Q}_3)$ again. There are essentially two (or three) different possibilities: vertices or the points on the middle of the edges. There are two different types of vertices, correspond to two different conjugacy classes of maximal compact subgroups, which are

$$x_1 \leftrightarrow \begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ \mathbb{Z}_3 & \mathbb{Z}_3 \end{bmatrix}, \qquad x_2 \leftrightarrow \begin{bmatrix} \mathbb{Z}_3 & 3\mathbb{Z}_3 \\ \frac{1}{3}\mathbb{Z}_3 & \mathbb{Z}_3 \end{bmatrix}.$$

For the point y in the middle of x_1 and x_2, it corresponds to

$$y \leftrightarrow \begin{bmatrix} \mathbb{Z}_3 & 3\mathbb{Z}_3 \\ \mathbb{Z}_3 & \mathbb{Z}_3 \end{bmatrix}.$$

The corresponding Moy–Prasad filtrations are given by

$$\begin{array}{ccc}
G_{x_1,0} = \begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ \mathbb{Z}_3 & \mathbb{Z}_3 \end{bmatrix} & G_{y,0} = \begin{bmatrix} \mathbb{Z}_3 & 3\mathbb{Z}_3 \\ \mathbb{Z}_3 & \mathbb{Z}_3 \end{bmatrix} & G_{x_2} = \begin{bmatrix} \mathbb{Z}_3 & 3\mathbb{Z}_3 \\ \frac{1}{2}\mathbb{Z}_3 & \mathbb{Z}_3 \end{bmatrix} \\
 & \cup & \\
\cup & G_{y,0.5} = \begin{bmatrix} 1+3\mathbb{Z}_3 & 3\mathbb{Z}_3 \\ \mathbb{Z}_3 & 1+3\mathbb{Z}_3 \end{bmatrix} & \cup \\
 & \cup & \\
G_{x_1,1} = \begin{bmatrix} 1+3\mathbb{Z}_3 & 3\mathbb{Z}_3 \\ 3\mathbb{Z}_3 & 1+3\mathbb{Z}_3 \end{bmatrix} & G_{y,1} = \begin{bmatrix} 1+3\mathbb{Z}_3 & 3^3\mathbb{Z}_3 \\ 3\mathbb{Z}_3 & 1+3\mathbb{Z}_3 \end{bmatrix} & G_{x_2,1} = \begin{bmatrix} 1+3\mathbb{Z}_3 & 3^3\mathbb{Z}_3 \\ 3\mathbb{Z}_3 & 1+3\mathbb{Z}_3 \end{bmatrix} \\
 & \cup & \\
\cup & G_{y,1.5} = \begin{bmatrix} 1+3^3\mathbb{Z}_3 & 3^3\mathbb{Z}_3 \\ 3\mathbb{Z}_3 & 1+3^3\mathbb{Z}_3 \end{bmatrix} & \cup \\
\vdots & \vdots & \vdots
\end{array}$$

They proved that any (smooth) representation of $G(F)$ possesses a vector fixed by a group in the filtration.

Theorem 3.2 (Moy–Prasad [15, Theorem 5.2])**.** If (π, V) is a smooth representation of G, then there is a nonnegative real number $r = \varrho(\pi)$ with the property that r is the minimal number such that $V^{G_{x,r^+}} \neq \{0\}$ for some $x \in \mathcal{B}(G)$.

We call the number $\varrho(\pi)$ as the *depth* of π. Espeically, depth-zero representation (π, V) is a representation with $V^{G_x^+} \neq \{0\}$. Note that the depth of a representation does not need to be an integer: we will see an example of depth $\frac{1}{2}$ representation in Section 4.

Moreover, they proved that all the depth-zero supercuspidal representations arise from the representations of finite groups, generalizing the construction of representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ in Section 1.

Definition 3.3. A *depth-zero minimal K-type* is a pair (G_x, τ) where $x \in \mathcal{B}(G)$ and τ is a cuspidal representation of the finite group G_x/G_x^+, inflated to G_x.

Theorem 3.4 (Moy–Prasad [15, 16])**.** Let (G_x, σ) be a minimal K-type and $\mathcal{E}(\sigma)$ be the set of irreducible representations of $F_x = N_G(G_x)$ (up to equivalence) whose restriction to G_x contains σ. Then for any $\tau \in \mathcal{E}(\sigma)$, $\mathrm{Ind}_{F_x}^G(\tau)$ an irreducible supercuspidal representation of G (necessarily of depth-zero), and any depth-zero irreducible supercuspidal representations arises from some $x \in \mathcal{B}(G)$ in this way.

The construction of the cuspidal representations of "finite groups of Lie type" (e.g. has a form of $\mathcal{G}(\mathbb{F}_q)$ for some $\mathcal{G}_{/\mathbb{F}_q}$) are usually understood by the Deligne–Lusztig theory (ℓ-adic cohomology of a variety $X_{/\mathbb{F}_q}$ with $\mathcal{G}(\mathbb{F}_q)$-action on it).

4 Yu's construction

As a follow-up of Theorem 3.4, it is natural to ask how to construct supercuspidal representations of positive depths. Following the folklore conjecture, our goal is to generalize the construction of supercuspidal representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ mentioned above to general reductive groups, via the following diagram:

$$\begin{array}{ccc} G(F) & & \\ \uparrow & & \\ K & \twoheadrightarrow & \mathcal{G}(\mathbb{F}_q) \end{array}$$

In other words, our goal is to find *some* compact-mod-center open subgroup K of $G(F)$ and *some* representation ρ of K factors through a Lie group of finite type $\mathcal{G}(\mathbb{F}_q)$, such that the compact induction $\mathrm{cInd}_K^{G(F)}\rho$ is supercuspidal.

Using Moy–Prasad filtration, Yu constructed tons of supercuspidal representations of reductive groups. Yu's data consists of the following ingredients:

Definition 4.1. *Yu's datum* is a tuple

$$((G_i)_{1\le i\le n+1}, x, (r_i)_{1\le i\le n}, \rho, (\psi_i)_{1\le i\le n})$$

for some $n \in \mathbb{Z}_{\ge 0}$, where

- $G = G_1 \supsetneq G_2 \supsetneq G_3 \supsetneq \cdots \supsetneq G_{n+1}$ are *twisted Levi subgroups* of G [5, Definition 4.1.1], which split over a tamely ramified extension of F,
- $x \in \widetilde{\mathcal{B}}(G_{n+1}) \subset \widetilde{\mathcal{B}}(G)$,
- $r_1 > r_2 > \cdots > r_n > 0$ are real numbers,
- ρ is an irreducible representation of $(G_{n+1})_{[x]}$ trivial on $(G_{n+1})_{x,0+}$, i.e. a depth-zero representation,
- ψ_i is a character of G_{i+1} of depth r_i,

satisfying the following conditions:

- $Z(G_{n+1})/Z(G)$ is anisotropic,
- the image $[x]$ of the point x in $\mathcal{B}(G_{n+1})$ is a vertex,
- $\rho|_{(G_{n+1})_{x,0}}$ is a supercuspidal representation of $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$,
- ψ_i is (G_i, G_{i+1})-*generic* relative to x of depth r_i [5, Definition 4.1.3] for all $1 \le i \le n$ with $G_i \ne G_{i+1}$.

Here $\widetilde{\mathcal{B}}(G)$ is the enlarged (non-reduced) buliding,

$$\widetilde{\mathcal{B}}(G) = \mathcal{B}(G) \times (X_\bullet(G) \otimes_{\mathbb{Z}} \mathbb{R}).$$

Theorem 4.2 (Yu [19])**.** Let $((G_i)_{1\le i\le n+1}, x, (r_i)_{1\le i\le n}, \rho, (\psi_i)_{1\le i\le n})$ be a Yu's datum. Define a compact-mod-center open subgroup K of $G(F)$ as

$$K = (G_1)_{x,\frac{r_1}{2}}(G_2)_{x,\frac{r_2}{2}}\cdots(G_n)_{x,\frac{r_n}{2}}(G_{n+1})_{[x]}$$

and $\tilde{\rho} = \rho \otimes \kappa$ be a representation of K, where ρ is trivially extended from $(G_{n+1})_{[x]}$ and κ is a certain representation which is built out of ψ_i's via Heisenberg representation (See [4, Section 3.8]). Then $\mathrm{cInd}_K^{G(F)} \tilde{\rho}$ is a supercuspidal smooth irreducible representation of $G(F)$ of depth r_1.

Intuitively, you have a depth-zero supercuspidal representation of the smallest group in the filtration (which are "multiplicative"), and enlarge it to a representation of $G(F)$ by using "additive" characters ψ_i.

For example, consider $G_1 = G = \mathrm{SL}_2(\mathbb{Q}_3)$ again. Take $n = 1$. Let G_2 be the non-split torus given by

$$G_2(F') = \left\{ \begin{bmatrix} a & b \\ 3b & a \end{bmatrix} \in \mathrm{SL}_2(F') : a, b \in F' \right\}$$

for all field extensions $F'/\mathbb{Q}_3$, which splits over a tame extension $\mathbb{Q}_3(\sqrt{3})$. Let x be the unique point of $\widetilde{\mathcal{B}}(S, \mathbb{Q}_3) \subset \widetilde{\mathcal{B}}(G, \mathbb{Q}_3)$. Let $r_1 = \frac{1}{2}$ and define $\psi_1 : G_2(\mathbb{Q}_3) \longrightarrow \mathbb{C}^\times$ as

$$\psi_1\left(\begin{bmatrix} a & b \\ 3b & a \end{bmatrix} \right) = \varphi(2b)$$

for a fixed additive character $\varphi : \mathbb{Q}_3 \to \mathbb{C}^\times$, which is nontrivial on $\mathbb{Z}_3$ but trivial on $3\mathbb{Z}_3$. Take ρ to be the trivial representation of $G_2(\mathbb{Q}_3) = (G_2)_{[x]}$. Then $((G_1, G_2), x, r_1, \rho, \psi_1)$ satisfies the conditions of Yu's construction, and it produces a supercuspidal representation of depth $\frac{1}{2}$. In this case, the compact subgroup is

$$K = \{\pm 1\} G_{x, \frac{1}{4}} = \left\{ \pm \begin{bmatrix} 1 + 3\mathbb{Z}_3 & \mathbb{Z}_3 \\ 3\mathbb{Z}_3 & 1 + 3\mathbb{Z}_3 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Q}_3) \right\}$$

and $\tilde{\rho}$ is the character of K given by $\tilde{\rho}(\pm 1) = 1$ and

$$\tilde{\rho}\left(\begin{bmatrix} 1 + 3a & b \\ 3c & 1 + 3d \end{bmatrix} \right) = \varphi(b + c).$$

Note that $[x] = y \in \mathcal{B}(\mathrm{SL}_2(\mathbb{Q}_3))$ is the point on the middle of the edge of x_1 and x_2 in Section 2.

There was an error in the original proof [19] (due to a misprinted statement in [9]), which was corrected by Fintzen [5] later (a counter example to the original argument is also provided). Also, Fintzen, Kaletha, and Spice showed that including a quadratic twist restores the validity of the original argument [7].

Theorem 4.3 (Kim [12], Fintzen [6], Fintzen–Schwein [8]). Suppose that G splits over a tamely ramified field extension of F (not necessarily characteristic zero). Then every supercuspidal smooth irreducible representation of $G(F)$ arises from Yu's construction, i.e., via Theorem 4.2.

Kim proved the exhausitiveness for characteristic zero fields with *very large* residue characteristic (with no effective bounds) [12], and Fintzen extended the result to *fairly large* residue characteristics, i.e. when p does not divide the order of the Weyl group of G [6]. Very recently, Fintzen and Schwein removed the condition and proved the exhausitiveness for all p [8].

It is natural to ask when two Yu's data produce equivalent representations. Hakim and Murnaghan [10] proved that two data produce equivalent representations if and only if one can obtain one from the other by a sequence of elementary transformation, $G(F)$-conjugation and refactorization.

References

[1] Bernstein, J. N., Deligne, P., et al. Le "centre" de Bernstein. In *Representations of reductive groups over a local field* (1984), pp. 1–32.

[2] Bruhat, F., and Tits, J. Groupes réductifs sur un corps local: I. données radicielles valuées. *Publications Mathématiques de l'IHÉS 41* (1972), 5–251.

[3] Bump, D. *Automorphic forms and representations*. No. 55. Cambridge university press, 1998.

[4] Fintzen, J. Supercuspidal representations: construction, classification, and characters.

[5] Fintzen, J. Representations of p-adic groups. *Current developments in mathematics 2021*, 1 (2021), 1–42.

[6] Fintzen, J. Types for tame p-adic groups. *Annals of Mathematics 193*, 1 (2021), 303–346.

[7] Fintzen, J., Kaletha, T., and Spice, L. A twisted Yu construction, Harish-Chandra characters, and endoscopy. *Duke Mathematical Journal 172*, 12 (2023), 2241–2301.

[8] Fintzen, J., and Schwein, D. Construction of tame supercuspidal representations in arbitrary residue characteristic.

[9] Gérardin, P. Weil representations associated to finite fields. *Journal of Algebra 46*, 1 (1977), 54–101.

[10] Hakim, J., and Murnaghan, F. Distinguished tame supercuspidal representations. *International Mathematics Research Papers 2008* (2008), rpn005.

[11] Howe, R. Tamely ramified supercuspidal representations of GL_n. *Pacific Journal of Mathematics 73*, 2 (1977), 437–460.

[12] Kim, J.-L. Supercuspidal representations: an exhaustion theorem. *Journal of the American Mathematical Society 20*, 2 (2007), 273–320.

[13] Morris, D. W. Introduction to Bruhat-Tits buildings.

[14] Moy, A. Local constants and the tame langlands correspondence. *American Journal of Mathematics 108*, 4 (1986), 863–929.

[15] Moy, A., and Prasad, G. Unrefined minimal K-types for p-adic groups. *Inventiones Mathematicae 116* (1994), 393.

[16] Moy, A., and Prasad, G. Jacquet functors and unrefined minimal K-types. *Commentarii Mathematici Helvetici 71* (1996), 98–121.

[17] Rabinoff, J. The Bruhat-Tits building of a p-adic Chevalley group and an application to representation theory, 2003.

[18] Serre, J.-P. *Trees*. Springer Science & Business Media, 2002.

[19] Yu, J.-K. Construction of tame supercuspidal representations. *Journal of the American Mathematical Society 14*, 3 (2001), 579–622.