Reference: Rabinoff's Senior Thesis "The Bruhat-Tits building..."

Notation: F local field, $\mathcal{O}_F$ ring of integers, $\mathfrak{p} \subseteq \mathcal{O}_F$ max'l ideal, $k = \mathcal{O}_F/\mathfrak{p}$ residue field

G reductive gp/F, $T \subset G$ a fixed split max'l torus, W Weyl group. $N \supset T$ the normalizer of T in G

$\Phi \subset E$ root system of G. $\Delta \subset \Phi$ simple roots.

$\Phi^\vee \subset E^*$ coroot system.

Def: Spherical apt of G is the Euclidean space $A_s := E^*$

For $\alpha \in \Phi$, hyperplane $H_\alpha \subset A_s$ is defined as set of $v \in E^*$ s.t. $\alpha^\vee(v) = 0$.

Weyl chamber $D \subset A_s$ is a connected component of $A_s \setminus \bigcup_{\alpha \in \Phi} H_\alpha$.

Fact: $D \longleftrightarrow$ a choice of simple roots of Φ.

W acts simply transitively on Weyl chambers (important Thm)

For $\alpha \in \Phi$, let $\mathcal{X}_\alpha \subseteq G$ be the root subspace.

Def: For $x \in E^*$,

$P_x := \langle T, \mathcal{X}_\alpha \ : \ \alpha \in \Phi, \ \alpha(x) \geq 0 \rangle$

Motto: "facets" of A_s parametrize parabolic $\overbrace{\text{subgroups of } G \text{ containing}}^{\text{subgroups of } G \text{ containing a } B\text{-conj.}}$ T

(Weyl chambers $\longleftrightarrow$ Borel subgroups) (a Borel subgrp is a max'l soluble subgrp)

The big picture for $G = SL_3$: let $\alpha = e_1 - e_2$, $\beta = e_2 - e_3$

$\mathcal{X}_\alpha = \begin{pmatrix} 1 & * & \\ & 1 & \\ & & 1 \end{pmatrix}$,

$\mathcal{X}_\beta = \begin{pmatrix} 1 & & \\ & 1 & * \\ & & 1 \end{pmatrix}$, etc.

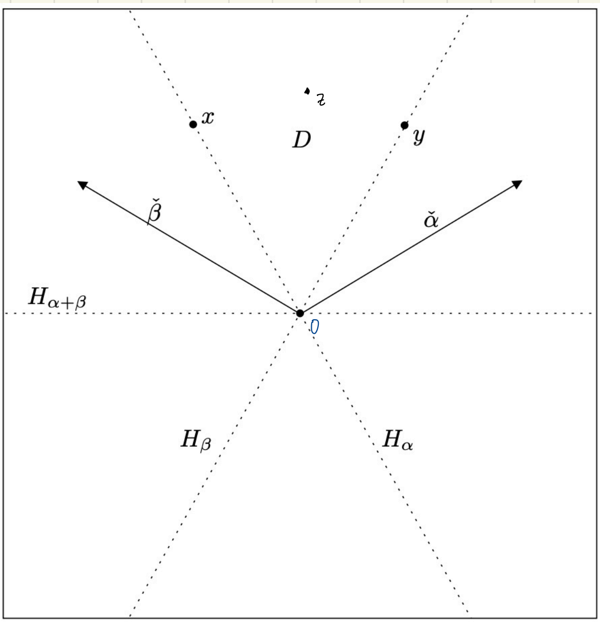

Figure 2.1: The spherical apartment of $\mathrm{SL}_3(\mathrm{f})$.

$P_z = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} = \langle T, \mathcal{X}_\alpha, \mathcal{X}_\beta, \mathcal{X}_{\alpha+\beta} \rangle$

$P_x = \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix} = \langle P_z, \mathcal{X}_{-\alpha} \rangle$

$P_y = \begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix} = \langle P_z, \mathcal{X}_{-\beta} \rangle$

$P_0 = SL_3$

Notice that

Def: The affine apartment of G is the affine space under A_s. So set theoretically A, A_s are the same.

A has no distinguished point (as affine space). But for convenience we do still distinguish the origin of E^*.

Thus $\{\text{point } p \in A\} \longleftrightarrow \{\text{vector } p - O \in A^s\}$

define an affine functional $\alpha + n : A \to \mathbb{R}$ by $(\alpha+n)(x) = \alpha(x - O) + n$

Hyperplane $H_{\alpha+n} \subset A$ is the "kernel" of $\alpha + n$. Let $s_{\alpha+n}$ be the reflection over $H_{\alpha+n}$.

Note $H_{\alpha+n} = H_{-\alpha-n}$.

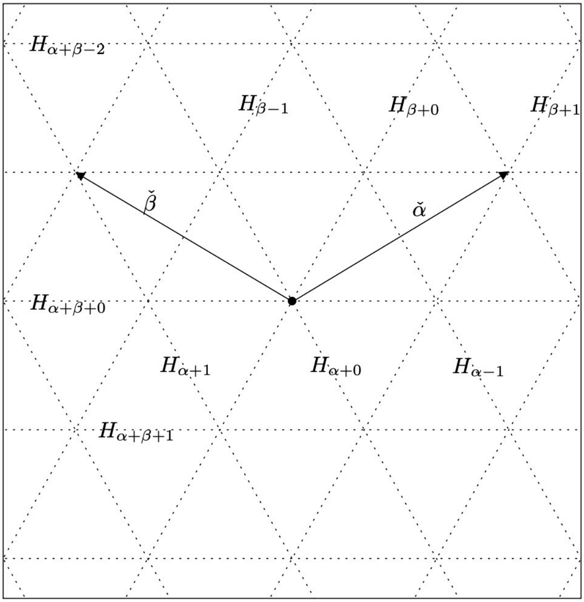

Figure 2.3: The affine apartment of $\mathrm{SL}_3(k)$.

Having defined the affine apartment, some questions: What is the analogue of the Weyl group? What about parabolic subgroups?

Def: The affine Weyl group $\widetilde{W}$ is the group of affine transformations of A generated by reflections $s_{\alpha+n}$.

Prop: $N(F)$ is the normalizer in $G(F)$ of $T(\mathcal{O}_F)$. And $\widetilde{W}$, $N(F)/T(\mathcal{O}_F)$ are both isomorphic to a semidirect product $\mathbb{Z}^{|\Delta|} \rtimes W$

In particular, the isomorphism is induced explicitly by a map $N(F) \longrightarrow \widetilde{W}$ that sends representatives of $N(F)/T(F)$ to the corresponding elt. of W, and $\check\alpha(t) \in T(F)$ to $T_\alpha^{-\|t\|}$

view $\check\alpha$ as a homomorphism $\mathbb{G}_m \to T$

where T_α is the translation $x \mapsto x + \check\alpha$ and $\|\cdot\|$ is the discrete valuation on t.

The map $N(F) \longrightarrow \widetilde{W}$ defines in an obvious way an action of $N(F)$ on the affine apartment A.

Example. Let $G = SL_2$. The affine apt is

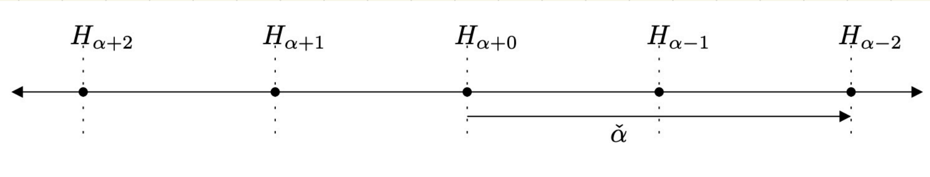

Figure 2.4: The affine apartment of $\mathrm{SL}_2(k)$.

$\check{\alpha}(\varpi^n) = \begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^{-n} \end{pmatrix} \longmapsto$ the translation which takes 0 to $H_{\alpha-2n}$.

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \longmapsto s_\alpha \in W$, which is the flip over $H_{\alpha+0}$.

Thus, $\begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^{-n} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is a flip over $H_{\alpha-n}$.

Definition of parahoric subgroups:

Let $\Psi = \{\alpha + n : \alpha \in \Phi, n \in \mathbb{Z}\}$ be the set of affine roots. For $\psi = \alpha + n \in \Psi$, write

$\mathcal{X}_\psi = \mathcal{X}_\alpha(\mathfrak{p}^n) = \{\mathcal{X}_\alpha(t) : t \in \mathfrak{p}^n\}$

For $x \in A$, define the group

$$G_x = \langle T(\mathcal{O}_F), \mathcal{X}_\psi : \psi \in \Psi, \psi(x) \geq 0 \rangle$$
$$= \langle T(\mathcal{O}_F), \mathcal{X}_\alpha(\mathfrak{p}^{-\lfloor \alpha(x) \rfloor}) : \alpha \in \Phi \rangle$$

A minimal G_x, i.e., x lies in a maximal facet, is called a parahoric subgroup.

Remark: a standard parahoric is given by

$$\begin{array}{ccc} G_x & \hookrightarrow & B(k) \\ \downarrow & & \cap \\ G(\mathcal{O}_F) & \twoheadrightarrow & G(k) \end{array}$$

The pullback of Borel subgroup wrt residue field.

Example: back to $G = SL_2$. See next page

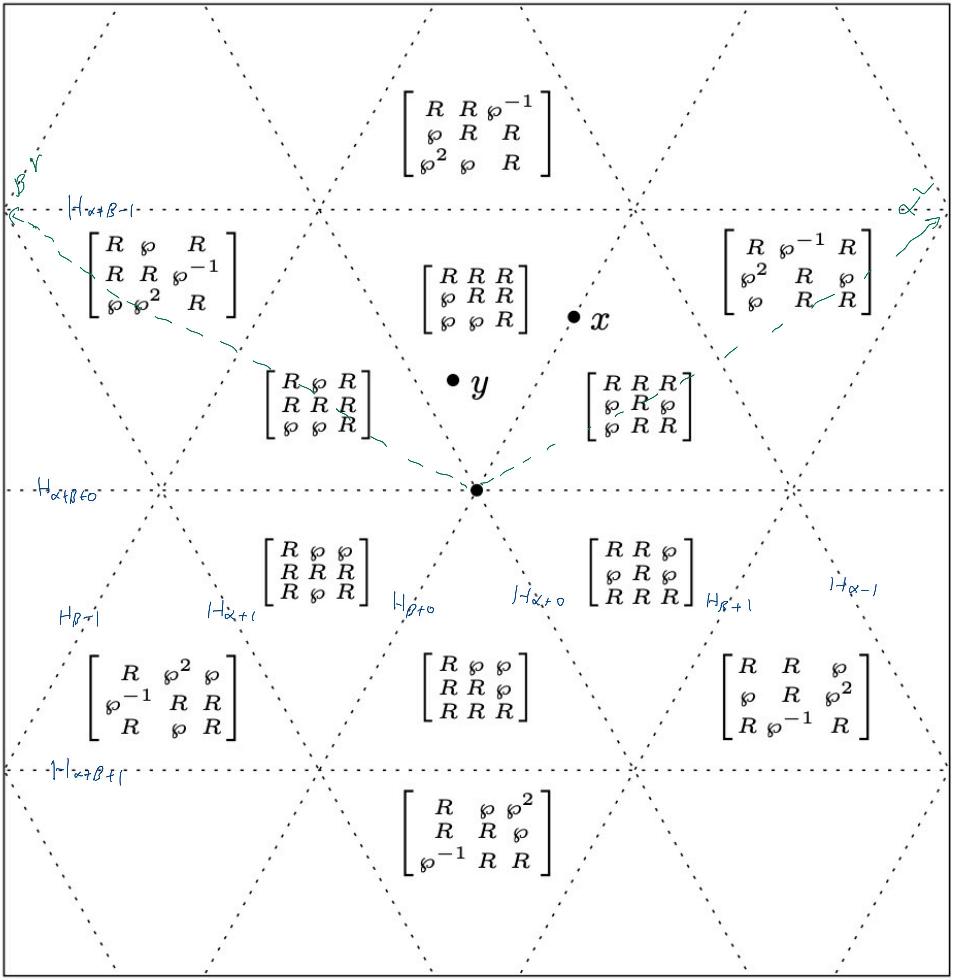

Figure 3.1: Some minimal parahoric subgroups of $\mathrm{SL}_3(k)$.

The point y is bounded between $H_{\alpha+0}$ and $\underset{= H_{-\alpha+1}}{H_{\alpha-1}}$, and ...

So $G_y = \langle T(\mathcal{O}_F), X_{\alpha+0}, X_{\beta+0}, X_{\alpha+\beta+0}, X_{-\alpha+1}, X_{-\beta+1}, X_{-\alpha-\beta+1} \rangle$

$$= \left\langle \begin{pmatrix} \mathcal{O}_F & & \\ & \mathcal{O}_F & \\ & & \mathcal{O}_F \end{pmatrix}, \begin{pmatrix} 1 & \mathcal{O}_F & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \mathcal{O}_F \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \mathcal{O}_F \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ \wp & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & \wp & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ \wp & & 1 \end{pmatrix} \right\rangle = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F & \mathcal{O}_F \\ \wp & \mathcal{O}_F & \mathcal{O}_F \\ \wp & \wp & \mathcal{O}_F \end{pmatrix}$$

Similarly,

$$G_x = \langle T(\mathcal{O}_F), X_{\alpha+0}, X_{-\alpha+1}, X_{\beta+0}, X_{-\beta+0}, X_{\alpha+\beta+0}, X_{-\alpha-\beta+1} \rangle$$

$$= \begin{bmatrix} \mathcal{O}_F & \mathcal{O}_F & \mathcal{O}_F \\ \wp & \mathcal{O}_F & \mathcal{O}_F \\ \wp & \mathcal{O}_F & \mathcal{O}_F \end{bmatrix}.$$

Remarks:

- Inside the "central hexagon," all of our parahorics are pullbacks of parabolics over k.
  (corresponds to the spherical apartment picture)
- For any facet with parahoric G_F assigned to that facet, then G_x is either the intersection of all parahorics corresponding to adjacent facets of smaller dimension, or is the grp generated by all parahorics corresponding to facets of larger dimension.
- Iwahoris are highest dimension facets, max'l parahorics are lowest dimension facets
  are also the max'l compact subgroups of $G(F)$
- Affine Weyl group $\tilde{W}$ acts simply transitively on the set of maximal facets
  - analogue of W acting simply transitively on the Weyl chambers
- Also, $\tilde{W}$ is generated by the affine root hyperplanes adjacent to any max'l facet.
- action of $N(F)$ on the affine apartment corresponds to conjugation of parahorics.

For the **Last point** let's return to the example

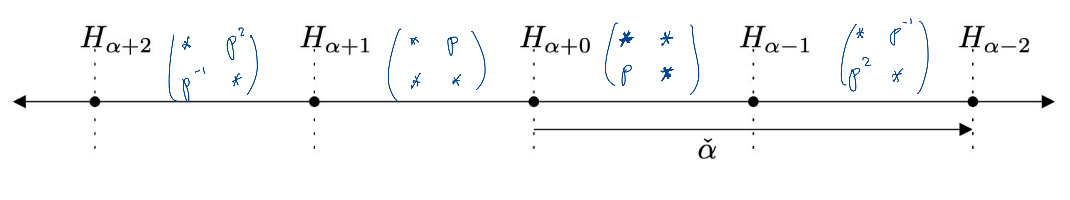

Figure 2.4: The affine apartment of $\mathrm{SL}_2(k)$.

Motto: Affine apartment parametrizes compact open subgroups of $G(F)$ containing an Iwahori (conjugate of I)