Bruhat-Tits building

Seewoo

Reference : Rabinoff Chap 4.

Recall that we (Brian) defined spherical / affine apartments as E^* with / without origin, for chosen max. torus T.

Affine apartment $\mathcal{A} = \mathcal{A}(G, T)$ has hyperplane structure

$$H_{\alpha+n} = \text{"}\ker(\alpha+n)\text{"} = \{x : \alpha(x-o) + n = 0\}$$

$\alpha \in \Phi$, $n \in \mathbb{Z}$ ($\alpha+n$ is called affine lin. functional)

Affine Weyl group $\widetilde{W}$ is generated by reflections over affine hyperplanes, isom. to $\mathbb{Z}^{|\Delta|} \rtimes W \simeq N_G(T)/T(\mathcal{O})$.

In other words, $N_G(T)$ acts on $\mathcal{A}$ through $\widetilde{W}$.

Parahoric subgroup is : for $x \in \mathcal{A}$,

$$G_x = \langle T(\mathcal{O}), \{\mathcal{X}_{\alpha+n} : (\alpha+n)(x) \geq 0, \alpha \in \Phi\}\rangle$$

$$= \langle T(\mathcal{O}), \{\mathcal{X}_\alpha(\mathfrak{p}^{-\lfloor \alpha(x) \rfloor}) : \alpha \in \Phi\}\rangle$$

where $\mathcal{X}_\alpha$ is root group (space) of α. $(\subset G)$.

(pro-) unipotent radical of G_x is defined

$$G_x^+ = \langle T(1+\mathfrak{p}), \{\mathcal{X}_{\alpha+n} : (\alpha+n)(x) > 0, \alpha \in \Phi\}\rangle$$

$$= \langle T(1+\mathfrak{p}), \{\mathcal{X}_\alpha(\mathfrak{p}^{1-\lceil \alpha(x) \rceil}) : \alpha \in \Phi\}\rangle.$$

$G_x^+ \subset G_x$, and $\mathcal{G}_x := G_x / G_x^+$ is always a Lie group of fin. type, which is important for Moy-Prasad filtration (it is "depth-0 piece" of it).

To define building, we want to remove dependency of T in $\mathcal{A}(G,T)$: glue all of them together!

<u>Def</u> Fix $\mathcal{A}$. Bruhat-Tits building of G/F is defined by

$$\mathcal{B}(G) := G \times \mathcal{A} / \sim$$

where equivalence relation is : $(g,x) \sim (h,y)$ iff $\exists\, n \in N = N_G(T)$ s.t. $y = nx$ & $g^{-1}hn \in G_x$.

- Why? Here's my explanation : in fact, we are not glueing G-many apartments. Each point in $\mathcal{A}$ is fixed by $T(\mathcal{O})$-action & $T(\mathcal{O}) \subset G_x$, the above $\sim$ is actually equivalent to

$$\exists\, w \in \widetilde{W} \simeq N_G(T)/T(\mathcal{O}) \text{ s.t. } y = wx \ \& \ g^{-1}hw \in G_x.$$

($w \in G$ by choosing a representative). In other words we are identifying x and y via $\widetilde{W}$-action, and consider all the identifications, but "part" of $\mathcal{A}$ only. (more explanation for SL_2 will come up later.)

- $\mathcal{B}(G)$ admits natural G-action : $g \cdot [h, x] := [gh, x]$.
- Each $g \in G$ gives $\mathcal{A} \hookrightarrow \mathcal{B}$. $x \longmapsto [g, x)$.
  Especially, we identify $\mathcal{A}$ with $1 \times \mathcal{A} \subset \mathcal{B}$.
- $\bigcup_{g \in G} g \cdot \mathcal{A} = \mathcal{B}$.
- G-action on $\mathcal{B}$ does not preserve $\mathcal{A}$ in general.
  In fact, $\mathrm{Stab}_G(\mathcal{A}) = N$.
- All maximal tori conjugate each other. For a new torus ${}^gT = gTg^{-1}$, define an apartment for it as
  $\mathcal{A}({}^gT) := g \cdot \mathcal{A}(T)$. $\mathcal{A}({}^gT) = \mathcal{A}({}^hT)$
  iff ${}^gT = {}^hT \iff g^{-1}h \in N$.
- Parahoric subgroups of $x \in \mathcal{B}(G)$ is : choose $g \in G$, $x_0 \in \mathcal{A}$ s.t. $x = gx_0$. Define $G_x^{(+)} := g\, G_{x_0}^{(+)}\, g^{-1}$.
- Facet of $\mathcal{B}$ = $g \cdot$ (facet of $\mathcal{A}$).
  Action is transitive on <u>maximal</u> facets.

One useful theorem on $\mathcal{B}(G)$:

<u>Thm</u> (Corollary 4.14) Any two points in $\mathcal{B}(G)$ is contained in an apartment $\mathcal{A}' \subset \mathcal{B}$.

This follows from (in fact equivalent?) affine Bruhat decomposition:

<u>Thm</u> (Theorem 3.36) $\forall x, y \in \mathcal{A}$, $G = G_x N G_y$.

We can impose a metric on $\mathcal{B}$ as follows:

<u>Def</u> Let $x, y \in \mathcal{B}$. Choose an apartment $\mathcal{A}' \subset \mathcal{B}$ containing both x and y. Let $g \in G$ s.t. $g \cdot \mathcal{A}' = \mathcal{A}$. Define $d(x, y) := \| g \cdot x - g \cdot y \|_{\mathcal{A}_s}$ (metric from E^*).

- Well-definedness: we need some lemmas.

<u>Lemma 1</u> (Lemma 4.9). $[g, x] \in \mathcal{A}$ iff $g \in N \cdot G_x$

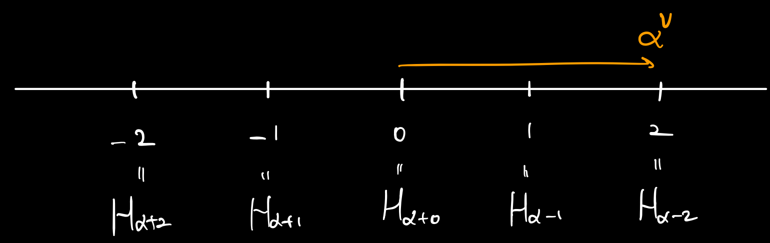
<u>Lemma 2</u> (Theorem 3.40) $\bigcap_{x \in \Omega} (N \cdot G_x) = N \cdot (\bigcap_{x \in \Omega} G_x)$

<u>Lemma 3</u> (Lemma 4.19) $x, y \in \mathcal{B}(G)$. Let $\mathcal{A}_1, \mathcal{A}_2$ be two apartments containing x, y. Then the line segment $\overline{xy}$ is also contained in both, and length of it does not depend on choice of $\mathcal{A}_1$ or $\mathcal{A}_2$.

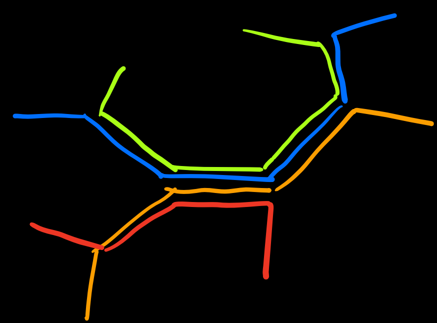
- Triangle inequality: not easy (see original [BT])

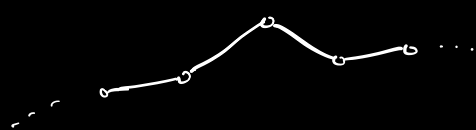
Example : $SL_2(F)$

$\mathcal{A} \simeq \mathbb{R}$.



We are glueing bunch of lines $\Rightarrow$ gives a graph. "homogenous".



Define a path in obvious way. 
(line).

Thm (Lemma 4.25) Any line is contained in an apartment.
In particular, there's no cycle $\Rightarrow$ $\mathcal{B}(SL_2)$ is a tree.

Thm (Lemma 4.26) Any vertex has order $q+1$.

pf) Assume $x = 0 \in \mathcal{A}$ is the origin. If e is an edge with one end x, then $G_x = SL_2(\mathcal{O}) = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} \supset G_e = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix}$.
Then edges adjoining x is in bijection with G_x / G_e:
if $g \cdot e$ adjoins x, we have $g \cdot x = x$ (pf of Lemma 4.25) and $g \in G_x$. Conversely, $g \cdot e = h \cdot e$ iff $g^{-1}h \in G_e$. Now $G_x / G_e \simeq \mathbb{P}^1(\mathbb{F}_q)$. ∎

Compare w/ another defn in Serre "Trees".

$\mathcal{B}(SL_2(F))$ is a tree, where

- vertices : isomorphism (i.e. homothety) class of lattices in $F^{\oplus 2}$ (i.e. free $\mathcal{O}_F$-submodule)

- edges : $[L]$ and $[L']$ connected if $L' \subset L$ and $L/L' \simeq (\mathbb{F}_q, +)$.

Natural action : $g \in SL_2(F)$, $g \cdot [L] := [g \cdot L]$

Metric : For any L, L', $\exists$ basis $\{v_1, v_2\}$ of L s.t.
$L' = \varpi^a v_1 \oplus \varpi^b v_2$, $a \geq b \in \mathbb{Z}$. Then $d([L],[L']) := a-b$.

$SL_2(F)$-invariant.

This is in bijection w/ above definition

. "standard" apartment. $\mathcal{A} \simeq 1 \times \mathcal{A} \subset \mathcal{B}$

$\Longleftrightarrow$ lattices of the form $\varpi^a e_1 \oplus \varpi^b e_2$, $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$T(F) = \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \right\}$. acts as translation. (by even)

$N(T) = \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} & * \\ * & \end{pmatrix} \right\}$ translation & reflection (by even).