Buildings 2.0

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All references are to Kaletha–Prasad (except one Rabinoff diagram).

Start field k, connected reductive k-group G/k. $\qquad G_{der} := [G,G]$

Assume G split for now, but later will consider quasisplit case. All the definitions of subgroups below are general except for $G(k)^0$, which has a case-by-case definition (I only wrote the quasisplit definition) and $T(k)^0$ (which I didn't give the full quasisplit definition for).

$$\begin{array}{ccccc} & & G & & \\ & & \cup & & \\ G_{sc} & \twoheadrightarrow & G_{der} & \twoheadrightarrow & G_{ad} \end{array}$$

Take S split max'l torus in G (max'l for split G). $S_{der} = S \cap G_{der}$, S_{sc} = preimage, S_{ad} = image.

Have bijections between $\Phi(G_{sc}, S_{sc})$, $\Phi(G,S)$, $\Phi(G_{der}, S_{der})$, $\Phi(G_{ad}, S_{ad})$. §2.6(a)

Def: Root system $\Phi(G,S)$ is subset of $\mathbb{R} \otimes_{\mathbb{Z}} X^*(S)$ consisting of nonzero weights for action of S on $\mathrm{Lie}(G)$. §2.6(b)

not sure if this is used

Now assume k Henselian with nontrivial discrete valuation $\omega : k^\times \to \mathbb{R}$, $\mathcal{O} = \omega^{-1}(\mathbb{R}_{\geq 0})$, $\mathfrak{f} = \mathcal{O}/\mathfrak{m}$ (perfect?)

Associated to S is (non-enlarged) apartment $\mathcal{A}(S)$, which is affine space based on $\mathbb{R} \otimes_{\mathbb{Z}} X_*(S_{der})$

Have bijections between $\mathcal{A}(S_{sc})$, $\mathcal{A}(S)$, $\mathcal{A}(S_{der})$, $\mathcal{A}(S_{ad})$. §4.1, Axiom 4.1.1

Same with buildings. So in particular, action of $G(k)$ on $\mathcal{B}(G)$ factors through $G_{ad}(k)$.

How to think of apartments & construct parahoric subgps?

Def of bounded subschemes: §2.2

Def: $G(k)^1 = \{ g \in G(k) \mid \chi(g) \in \mathcal{O}^\times \ \forall \chi \in X^*_k(G) \}$

Prop 2.2.16

this is S in split case, and max'l torus T in quasisplit case

$$G(k)^b = G(k)^\natural \cdot Z(k)^1 = G(k)^0 \cdot Z(k)^1, \text{ where } Z := Z_G(S)$$

§2.6(e) $G(k)^\natural = \mathrm{im}(G_{sc}(k) \to G(k))$

$G(k)^0 = G(k)^\natural \cdot T(k)^0$ uses quasi-split: $T := Z_G(S) := S$ for split 2.5(d)

In general, we have $G(k)^\natural \subset G(k)^0 \subset G(k)^b \subset G(k)^1$ (finite index)

Rmk: $S(k)^0 = S(k)^1$ for S split, or more generally, if $S_{k^{ur}}$ is induced. The latter is true if S is induced, which holds for S a maximally split maximal torus in a quasisplit group of simply-connected or adjoint type. 2.5(a)

Rmk: Why is $G(k)^\natural \neq G_{der}(k)$? Consider the sequence $1 \to \mu_n \to SL_n \to PGL_n \to 1$.

Taking étale coh. gives
$$1 \to \mu_n(k) \to SL_n(k) \to PGL_n(k) \to \underbrace{H^1(k, \mu_n)}_{= k^\times/(k^\times)^n} \to \underbrace{H^1(k, SL_n(k))}_{= 0}.$$

Ex: $\begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}$ is in $PGL_2(k)$ but not in the image of $SL_2(k)$.

Now we can make a table with groups of interest for us (everything is split here).

G	$G(k)'$	$G(k)^b$	$G(k)^0$	$G(k)^\natural$
GL_n/k	$SL_n(k)\cdot\mathcal{O}^\times \mathrm{id}$	"	"	$SL_n(k)$
SL_n/k	$SL_n(k)$	"	"	"
PGL_n/k	$PGL_n(k)$	"	index n subgp $= \mathrm{im}(SL_n(k)\to PGL_n(k))$	"
split torus S/k	"$S(\mathcal{O})$" $= \begin{pmatrix}\mathcal{O}^\times & & \\ & \mathcal{O}^\times & \\ & & \ddots & \\ & & & \mathcal{O}^\times\end{pmatrix}$	"	"	"

Now we can define the parahoric subgroups and other bounded subgroups associated to an apartment. We still work in the split case.

We have root unipotent subgps U_a for $a\in\Phi$, with $U_a \cong \mathbb{G}_a$ non-canonically.

So the groups U_a are canonically isomorphic in G, G_{der}, G_{sc}, G_{ad}.

If we fix a Chevalley System, i.e. a compatible choice of $u_a : \mathbb{G}_a \xrightarrow{\sim} U_a$ (equivalent to choosing element $du_a(1) = X_a \in \mathfrak{g}_a$)

then we get a basepoint $0 \in A(S)$

Given $x\in A(S)$, we construct a parahoric $P_x = \langle S(k)^0, \underbrace{u_a(\mathfrak{m}^{-\lfloor a(x)\rfloor})}_{\text{Version of } U_a \text{ but only for } \mathfrak{m}^{-\lfloor a(x)\rfloor} \text{ instead of } k} \rangle \subset G(k)^0$. §6.1

Both u_a and $a(x)$ depend on the basepoint, and they cancel out so that P_x depends only on the abstract point x.

Using the $G(k)$-conjugation action and an equivalence relation, we can construct the building $B(G)$.

Then P_x is also $G(k)^0$-stabilizer. Thus $P_x = N_{G(k)^0}(P_x)$. So we can write $P_x = G(k)^0_x$.

For more info on this fact, see the last paragraph to the introduction of Chapter 7. The above description is the "by hand" construction, and there is also a construction by looking at the stabilizer under the conjugation action. They agree by Prop 7.6.4.

In general, we let $G(k)^*_x$ be the stabilizer of $x\in B(G)$ by the action of $G(k)^*$, where $* = 0, 1, b,$ or nothing. §7.6, 7.7

By Prop 7.6.4, if Ω lies in a facet, then the stabilizer and pointwise stabilizer of Ω in $G(k)^0$ coincide. Prop 7.7.5

Exp: In general, this is not true for $G(k)'$. Consider

$$\begin{tikzcd} \begin{bmatrix}\mathcal{O} & \mathcal{O}\\ \mathcal{O} & \mathcal{O}\end{bmatrix} \arrow[r, dash, "\begin{bmatrix}\mathcal{O} & \mathfrak{m}\\ \mathcal{O} & \mathcal{O}\end{bmatrix}"] & \begin{bmatrix}\mathcal{O} & \mathfrak{m}\\ \mathfrak{m}^{-1} & \mathcal{O}\end{bmatrix} \end{tikzcd}$$

, the standard chamber for $PGL_2(k)$.

Then $\begin{bmatrix} & \varpi\\ 1 & \end{bmatrix}\begin{bmatrix}\mathcal{O} & \mathcal{O}\\ \mathcal{O} & \mathcal{O}\end{bmatrix}\begin{bmatrix} & 1\\ \varpi^{-1} & \end{bmatrix} = \begin{bmatrix} & \varpi\\ 1 & \end{bmatrix}\begin{bmatrix}\mathfrak{m}^{-1} & \mathcal{O}\\ \mathfrak{m}^{-1} & \mathcal{O}\end{bmatrix} = \begin{bmatrix}\mathcal{O} & \mathfrak{m}\\ \mathfrak{m}^{-1} & \mathcal{O}\end{bmatrix}$,

$\begin{bmatrix} & \varpi\\ 1 & \end{bmatrix}\begin{bmatrix}\mathcal{O} & \mathfrak{m}\\ \mathfrak{m}^{-1} & \mathcal{O}\end{bmatrix}\begin{bmatrix} & 1\\ \varpi^{-1} & \end{bmatrix} = \begin{bmatrix} & \varpi\\ 1 & \end{bmatrix}\begin{bmatrix}\mathcal{O} & \mathcal{O}\\ \mathfrak{m}^{-1} & \mathfrak{m}\end{bmatrix} = \begin{bmatrix}\mathcal{O} & \mathcal{O}\\ \mathcal{O} & \mathcal{O}\end{bmatrix}$.

So $\begin{bmatrix} & \varpi\\ 1 & \end{bmatrix}$ stabilizes this chamber, but not pointwise.

So for a facet F, we define $G(k)'_F \subset G(k)^+_F$, being pointwise stabilizer (stabilizer of F in $G(k)'$.

Warning: The book also defines $G(k)^\sharp_\Omega$, but this doesn't have an interpretation in terms of stabilizers.

$$\begin{bmatrix}\mathcal{O} & m^{-1}\\ m & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & \mathcal{O}\\ m & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & \mathcal{O}\\ \mathcal{O} & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & m\\ \mathcal{O} & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & m\\ m^{-1} & \mathcal{O}\end{bmatrix}$$

Gives $SL_2(k)_F^1 = SL_2(k)_F^+ = SL_2(k)_F^0 = SL_2(k)_F$: intersect with $SL_2(k)$

$$\begin{bmatrix}\mathcal{O} & m^{-1}\\ m & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & \mathcal{O}\\ m & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & \mathcal{O}\\ \mathcal{O} & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & m\\ \mathcal{O} & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & m\\ m^{-1} & \mathcal{O}\end{bmatrix} \Bigg|_{\det = 1}$$

Gives $PGL_2(k)_F^1 = PGL_2(k)_F^0$: image in $PGL_2(k)$

$$\begin{bmatrix}\mathcal{O} & m^{-1}\\ m & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & \mathcal{O}\\ m & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & \mathcal{O}\\ \mathcal{O} & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & m\\ \mathcal{O} & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & m\\ m^{-1} & \mathcal{O}\end{bmatrix}$$

Gives $PGL_2(k)_F = PGL_2(k)_F^+ = PGL_2(k)_x^1$: image in $PGL_2(k)$

for x midpoint of chamber

Rmk: For x not a chamber midpoint, $PGL_2(k)_x^0 = PGL_2(k)_x^1$

$$\begin{bmatrix}\mathcal{O} & m^{-1}\\ m & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & \mathcal{O}\\ m & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & \mathcal{O}\\ \mathcal{O} & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & m\\ \mathcal{O} & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & m\\ m^{-1} & \mathcal{O}\end{bmatrix} \Bigg|_{\det \in \mathcal{O}^\times}$$

Gives $GL_2(k)_F^+ = GL_2(k)_F^1 = GL_2(k)_F^0$.

$$\begin{bmatrix}\mathcal{O} & m^{-1}\\ m & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & \mathcal{O}\\ m & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & \mathcal{O}\\ \mathcal{O} & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & m\\ \mathcal{O} & \mathcal{O}\end{bmatrix} \quad \begin{bmatrix}\mathcal{O} & m\\ m^{-1} & \mathcal{O}\end{bmatrix}$$

gives $GL_2(k)_F$

Rmk: for x not a midpoint in the interior of a chamber, $GL_2(k)_x = GL_2(k)_x^0 \cdot k$ id $= GL_2(k)_F \mid \det$ has even valuation

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Enlarged building. Use $\mathbb{R} \otimes_{\mathbb{Z}} X_*(T)$ instead of $\mathbb{R} \otimes_{\mathbb{Z}} X_*(T_{der})$.

Now the center of $G(k)$ acts on the building nontrivially.

Rmk. $\begin{pmatrix} & \varpi \\ 1 & \end{pmatrix}$ acts on $\tilde{B}(GL_2/k)$ by a glide reflection.

Overall remarks: All the $G(k)_F^{1,+,0}$ are bounded open subgroups of $G(k)$.

- In general, $G(k)_F^+ = G(k)^1_{\text{barycenter of } F}$. So $G(k)_x^1$ depends on x, but $G(k)_x^0 = G(k)_F^0$ for any $x \in F$.
- All bounded open subgroups of $G(k)$ lie in $G(k)^1$.
- $G(k)_F^0 \subset G(k)_F^1 \subset G(k)_F^+ \subset G(k)_F$. $G(k)_F$ is not bounded in general.
- $G(k)_x$ can have weird behavior (similar to $G(k_x)$), since e.g. $\begin{bmatrix} & \varpi \\ 1 & \end{bmatrix}$ flips the chamber orientation.
- $G(k)_x^\#$ also exists but I haven't defined it. It is a subgroup of $G(k)_x^0$.
- $G(k)_x^\#$ is not $G(k)_{x,0^+}$ in general, since for $F_1 \subset \bar{F}_2$, we have $G(k)_{F_2}^\# \subset G(k)_{F_1}^\#$, but $G(k)_{F_1,0^+} \subset G(k)_{F_2,0^+}$ (see below).

Moy-Prasad subgroups In general, defining the Moy-Prasad subgroups of parahoric subgroups requires us to have a filtration on the unipotent groups $U_a(k)$ at a point $x \in A(S)$, as well as on $S(k)$ itself.

The $\mathbb{N}$-indexed filtration on U_a is canonical (Def 6.1.2) but there is more freedom for the filtration on the torus (more specifically on $Z(k)$). For the split and quasi-split cases, Z is a torus T and there is a canonical filtration given below.

Def (Moy-Prasad):

$$T(k)_r := \{ t \in T(k)_0 \mid \omega(\chi(t)-1) \geq r \quad \forall \chi \in X^*(T) \}$$

e.g. $\begin{pmatrix} 1+\pi m^r & & \\ & \ddots & \\ & & 1+\pi m^r \end{pmatrix}$.

$$P_{x,0} := P_x .$$

$$P_{x,r} := \langle T(k)_r , U_a(\pi^{-[x(a)+r]}) \rangle$$

$$P_{x,r+} := \bigcup_{s>r} P_{x,s} .$$

Then $P_{x,0} = P_x$, $P_{x,0+}$ depend only on the facet of x. This is not true for $P_{x,r}$ in general.

$$P_{F_1,0^+} \subset P_{F_2,0^+} \subset P_{F_2,0} \subset P_{F_1,0}$$

For $F_1 \subset \bar{F}_2$, we have $P_{F_1,0^+} \subset P_{F_2,0^+} \subset P_{F_2} \subset P_{F_1}$.

from Rabinoff undergrad thesis

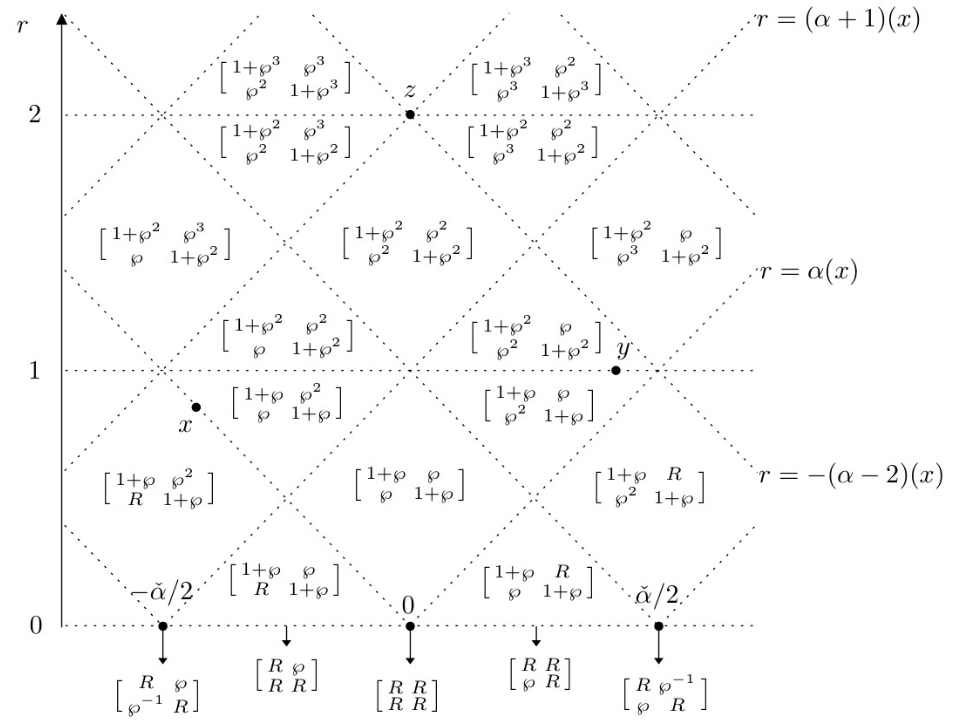


Figure 5.1: Some Moy-Prasad filtration subgroups for $SL_2(k)$. The affine apartment for $SL_2(k)$ is the bottom line in the figure.

Quasi-split case

Key property that makes this case nice: $Z_G(S) = T$, maximal torus.
S: maximally split torus

No anisotropic kernel: $[Z_G(S), Z_G(S)]$.

As all maximal split tori are $G(k)$-conj, all maximally split maximal tori are $G(k)$-conj. There is only one configuration (S,T) up to $G(k)$-conjugacy.

We have a relative root system $\Phi = \Phi(G,S)$, with all four versions the same.

The absolute root system comes from $\tilde{\Phi} = \Phi(G_{k_s}, T_{k_s})$. Let Θ be the Galois gp (action).

Then §2.6 $X^*(S) = X^*(T)_{\Theta,\text{free}}$, $X_*(S) = X_*(T)^{\Theta}$, and $\Phi \cup \{0\}$ is the image of $\tilde{\Phi}$ under the canonical map $X^*(T) \to X^*(S)$.

Two types of roots (and unipotent subgroups) for S.

Type 1: $\mathbb{R}_{>0} \cdot a \cap \Phi = a$. (non multipliable/divisible)

Then some number of roots in $\tilde{\Phi}$ map to Φ, with a transitive Θ-action.

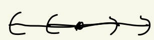
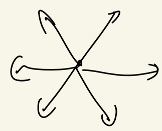
Take a representative $\tilde{a} \in \tilde{\Phi}$, so $U_{\tilde{a}} \cong \mathbb{G}_a$ over $k_{\tilde{a}}$, the fixed field of the Θ-action on $\tilde{a}$.

Then $U_a = \mathrm{Res}_{k_{\tilde{a}}/k} U_{\tilde{a}}$ is commutative.

Type 2/3: $\mathbb{R}_{>0} \cdot a \cap \Phi = \{a, 2a\}$.

Then have some number of triples $\tilde{a}, \tilde{a}', \tilde{b}$ with $\tilde{a} + \tilde{a}' = \tilde{b}$ mapping to $a, a, 2a$, respectively.

Then $U_a = \mathrm{Res}_{k_{\tilde{a}}/k} U_{[\tilde{a}]}$, $U_{[\tilde{a}]} = U_{\tilde{a}} \cdot U_{\tilde{a}'} \cdot U_{\tilde{b}}$ is a 3-dim. non-comm. unipotent group.

Ex: quasi-split $SU_3(k)$ has relative root system  coming from .
for quadratic extension ℓ/k

Generally, all the definitions are the same as above (first page of notes): $A(S)$ based on $\mathbb{R} \otimes_{\mathbb{Z}} X_*(S^{der})$, etc.

Only need to really figure out the unipotent group filtrations (on U_a).

See §3.2 and §6.1(b).