Hands in the 5-cards poker and their probabilities

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There are 10 kinds of *hands* in the 5-card poker. The following table shows frequency and probability for each hands.

Hand	Frequency	Probability
Royal flush	$\binom{4}{1} = 4$	0.000154%
Straight flush	$\binom{9}{1}\binom{4}{1} = 36$	0.00139%
Four of a kind	$\binom{13}{1} \cdot \binom{12}{1} \cdot \binom{4}{1} = 624$	0.0240%
Full house	$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3744$	0.1441%
Flush	$\binom{13}{5}\binom{4}{1} - \binom{10}{1}\binom{4}{1} = 5108$	0.1965%
Straight	$\binom{10}{1}\binom{4}{1}^5 - \binom{10}{1}\binom{4}{1} = 10200$	0.3925%
Three of kind	$\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2 = 54912$	2.1128%
Two pairs	$\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1} = 123522$	4.7539%
One pair	$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3 = 1098240$	42.2569%
No pair	$\left[\binom{13}{5} - 10\right]\left[\binom{4}{1}^5 - 4\right] = 1302540$	50.1117%

Now we will try to figure out how to compute these probabilities. First, we know that the probability space has

$$\binom{52}{5}$$

elements, since we are choosing 5 cards from a deck of 52 cards. (Order doesn't matter.) So we will count the number of possible cases for each hand and divide them by this number.

Royal flush

Royal flush is a set of 5 cards that contains 10, J, Q, K, A, and shape of the cards are same. For example,

10 ♦ J ♦ Q ♦ K ♦ A ♦

is a Royal flush.

Since numbers are fixed, we only need to choose shapes for 5 cards, and there are only 4 choices - heart, diamond, clover, and spade. So there are

$$\binom{4}{1} = 4$$

Royal flushes and the probability is

$$\frac{4}{\binom{52}{5}} \approx 0.00000154.$$

Straight flush

Straight flush is a set of 5 cards with consecutive numbers, from 1, 2, 3, 4, 5 to 9, 10, J, Q, K, and shape of the cards are same. We exclude 10, J, Q, K, A since it is a Royal flush. For example,

3 ♥ 4 ♥ 5 ♥ 6 ♥ 7 ♥

is a Straight flush.

For numbers, we have 9 choices:

$$\begin{gathered} 1, 2, 3, 4, 5 \\ 2, 3, 4, 5, 6 \\ \vdots \\ 8, 9, 10, J, Q \\ 9, 10, J, Q, K \end{gathered}$$

and we have 4 choices for shapes. Since they are independent, we have to multiply those two numbers and we get

$$\binom{9}{1}\binom{4}{1} = 36$$

for the number of Straight flushes. So the probability is

$$\frac{\binom{9}{1}\binom{4}{1}}{\binom{52}{5}} \approx 0.0000139.$$

Four of a kind

Four of a kind is a set of 5 cards where four of them have same numbers, such as

5♥ 5♦ 5♣ 5♠ J♦

First, we choose a number for the 4-cards block. We have $\binom{13}{1} = 13$ choices for that since we have 13 kinds of numbers: from A to K. Then we don't need to care about their shapes since they should have all different shapes. For the rest of card, we have $\binom{12}{1}$ choices for number since one of 13 choices is already chosen by 4-cards block. We can choose any shapes for that last card, which have $\binom{4}{1}$ choices. Thus there are

$$\binom{13}{1}\binom{12}{1}\binom{4}{1} = 624$$

possible Four of a kind and the probability is

$$\frac{\binom{13}{1}\binom{12}{1}\binom{4}{1}}{\binom{52}{5}} \approx 0.00024$$

Full house

Full house is a set of 5 cards where three of them are same numbers, and the other two also have same numbers. For example,

2♥ 2♣ 2♠ 6♥ 6♣

is a Full house.

We have $\binom{13}{1} = 13$ choices for the number of first 3-cards set, and $\binom{4}{3} = 4$ choices for their shapes. (They have 3 different shapes among 4 possible shapes.) For the other two cards, there are $\binom{12}{1} = 12$ choices for the number since one of 13 choices is already taken by the first set of 3-cards. For the shape, we have $\binom{4}{2} = 6$ choices since they have 2 different shapes among 4 possible shapes. By multiplying all of these, there are

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3744$$

possible Full houses and the probability is

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} \approx 0.001441.$$

Flush

Flush is a set of 5 cards where all the cards have same shapes, but not Royal flush or Straight flush. For example, A♠ 5♠ 7♠ J♠ K♠

is a Flush.

We have to choose 5 numbers among 13 possible numbers, which has $\binom{13}{5}$ ways to do. Also, since every card has same shape, we only need to choose one shape among 4 shapes, which is $\binom{4}{1} = 4$. However, we have to exclude Royal flush and Straight flush. So we have to subtract $4 + 36 = 40$ from the answer. Thus, there are

$$\binom{13}{5}\binom{4}{1} - 40 = 5108$$

possible Flushes and the probability is

$$\frac{\binom{13}{5}\binom{4}{1} - 40}{\binom{52}{5}} \approx 0.001965.$$

Straight

Straight is a set of 5 cards with consecutive numbers, but not all the same shapes (i.e. exclude Royal flush and Straight flush). For example,

3♠ 4♦ 5♠ 6♣ 7♥

is a Straight.

There are 10 possible choices for consecutive numbers: from 1, 2, 3, 4, 5 to 10, J, Q, K, A. Also, for each card we can choose any shape among 4 shapes, so there are $\binom{4}{1}^5 = 4^5$ choices for their colors. However, w can to exclude Royal flush and Straight flush. So we have to subtract $4 + 36 = 40$ from the answer as we did in Flush. Thus, there are

$$\binom{10}{1}\binom{4}{1}^5 - 40 = 10200$$

possible Straights and the probability is

$$\frac{\binom{10}{1}\binom{4}{1}^5 - 40}{\binom{52}{5}} \approx 0.003925.$$

Three of kind

Three of kind is a set of 5 cards where exactly three of them have same numbers. For example,

J♥ J♦ J♣ 4♥ 5♥

is a Three of kind.

First, we choose number among 13 numbers for the set of 3-cards, which has $\binom{13}{1} = 13$ ways to do. Then we have $\binom{4}{3} = 4$ choices for their shapes since they have 3 different shapes among 4 possible shapes. Now, for the numbers of last two cards, we have to choose two different numbers among 12 numbers that is not used yet, which has $\binom{12}{2}$ many ways to do. We can choose any shapes for that two cards, so we have $\binom{4}{1}^2 = 4^2$ for their shapes. By multiplying all of these numbers, there are

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{1}^2 = 54912$$

possible Three of kind and the probability is

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{1}^2}{\binom{52}{5}} \approx 0.021128.$$

Two pairs

Two pairs is a set of 5 cars which contains two pairs of same numbers, but the numbers for each pair should be different (i.e. not Four of a kind). Also, the other card's number should be different from the numbers of those pairs (i.e. not Full house). For example,

Q♥ Q♣ K♠ K♣ 7♠

is a Two pairs.

First, we choose two numbers for each pairs, which has $\binom{13}{2}$ possible choices. Then we choose shapes for each pairs. For a pair, the two cards in the pair should have different shapes among 4 shapes, which have $\binom{4}{2}$ ways to do. Since we have two pairs, we have to multiply $\binom{4}{2}$ for each pairs. Finally, the last card's number should be different from the numbers of those pairs, so we have $\binom{11}{1}$ choices for that, and $\binom{4}{1}$ choices for it's shape. By multiplying all of these numbers, there are

$$\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1} = 123552$$

possible Two pairs and the probability is

$$\frac{\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}}{\binom{52}{5}} \approx 0.047539.$$

One pair

One pair is a set of 5 cards which contains exactly one pair of two cards with same shapes. For example,

J♥ J♦ A♥ 3♠ 4♣

is a One pair.

First, we choose a number for the pair, which has $\binom{13}{1}$ possibilities, and shapes for the pair, which has $\binom{4}{2}$ possibilities. For the other three cards, their numbers should be all different and also different from the number of the pair, so we have to choose 3 different numbers among 12 numbers, which is $\binom{12}{3}$. There's no restriction on shapes of these 3 cards, so we have $\binom{4}{1}^3$ possibilities since we have $\binom{4}{1}$ many choices of shapes for each cards. By multiplying all of these numbers, there are

$$\binom{13}{2}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3 = 1098240$$

possible One pairs and the probability is

$$\frac{\binom{13}{2}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3}{\binom{52}{5}} \approx 0.422569.$$

No pair

No pair is a set of 5 cards which does not belong to any of the previous results. For example,

A♥ 2♥ 4♣ 7♣ J♠

is a No pair.

To find number of No pairs, we choose their numbers first. Since there's *no pair*, all of the numbers should be different, so we have to choose 5 different numbers among 13 numbers, which has $\binom{13}{5}$ many ways. However, we have to exclude the case of straight, which is formed with consecutive 5 numbers. There are 10 possible choices for consecutive numbers: from 1, 2, 3, 4, 5 to 10, J, Q, K, A. So we have to exclude those cases and we get $\binom{13}{5} - 10$ for numbers. For shapes, we can choose any shape for each card, so we get $\binom{4}{1}^5$. By the way, we have to exclude Flush, i.e. all of the cards have same shapes. For any given set of 5 different numbers, we have 4 different choices for shapes to be a Flush, and we have to exclude those cases. So we have $\binom{4}{1}^5 - 4$ for shapes. Now we multiply those two numbers (since they are independent) and we get

$$\left[\binom{13}{5} - 10\right]\left[\binom{4}{1}^5 - 4\right] = 1302540$$

for the number of No pairs, and the probability is

$$\frac{\left[\binom{13}{5} - 10\right]\left[\binom{4}{1}^5 - 4\right]}{\binom{52}{5}} \approx 0.501117.$$

What if...

Now assume that our Ramanujan is addicted to the 5-card poker and he played it 10000 times. What is a probability that Ramanujan get at least on Royal flush during the 10000 games? We can compute this by using the complement rule. The probability space can be divided into

$$\begin{aligned}\Omega =& \{\text{At least one Royal flush among 10000 games}\}\\ &\cup \{\text{No Royal flush among 10000 games}\}.\end{aligned}$$

So if A stands for the event that *Ramanujan got at least one Royal flush among 10000 games*, then

$$P(A) = 1 - P(A^c)$$

where A^c stands for the event that *Ramanujan got no Royal flush among 10000 games*. Now, for each game, the probability of *not* having Royal flush is

$$1 - 0.00000154 = 0.99999846$$

by complemet rule again, since the probability of having Royal flush is 0.00000154 as we find. If we (or Ramanujan) play 10000 games, then the probability of not having Royal flush for every 10000 game is

$$(1 - 0.00000154)^{10000} = 0.99999846^{10000}$$

since all the probabilities for each game are independent, so we can just multiply $1-0.00000154$ by 10000 times. Since the above number is the probability $P(A^c)$, we get

$$P(A) = 1 - P(A^c) = 1 - 0.99999846^{10000} \approx 0.015282 = 1.5282\%,$$

which is quite big (even bigger than the probability that Straight appears for a single game!). However, you need to play 10000 games to get this probability.

How many games do Ramanujan have to play to make the probability bigger than 50%? We need to find the smallest N such that

$$1 - \left(1 - \frac{4}{\binom{52}{5}}\right)^N \geq \frac{1}{2},$$

and it turns out that the smallest such N is $N = 450366$. So if you play poker 450366 games, then the probability to have at least one Royal flush is bigger than 50%! But, keep in mind that you have to play **450366** games.