MATH 10B with Prof. Stankova
DIS 102; TTh 8:00~ 9:30am
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# Quiz 8

Student:
SID:
Tue 3/19/19

**True/False** - No explanation needed. (2pts)

1. For any random variable $X$, we have $E[X^2] \geq E[X]^2$. **True**/False

   *sol.* We have $\mathrm{Var}[X] = E[X^2] - E[X]^2$, and variance of random variable is always non-negative.

2. According to the Law of Large Numbers, the probability $P(|\overline{X} - \overline{\mu}| > \epsilon)$ decreases as $n$ grows. True/**False**

   *sol.* As $n$ grows to $\infty$ the probability mass becomes more and more concentrated around the mean $\mu$. However, for specific $\epsilon$ and $n$ it may be the case that the probability increases slightly when we take the average of $n+1$ random variables rather than $n$.

**Problems** - Need justification. No justification means **zero**!

In this problem, we will approximate the following summation

$$\sum_{k=45}^{55} \binom{100}{k} \frac{1}{2^{100}}$$

using the standard normal distribution and the Central Limit Theorem. Let $X_1, \dots, X_{100}$ be the Bernoulli variables with $p = 1/2$. Let $\overline{X} = \frac{X_1 + \cdots + X_{100}}{100}$ be the average, and let $Z$ be the normalized RV corresponding to $\overline{X}$.

1. Find the range of $\overline{X}$, $\overline{\mu}$, and $\overline{\sigma}$. (3pts)

   *sol.* Since the range of each $X_i$s are $\{0, 1\}$, the range of $X_1 + \cdots + X_{100}$ is $\{0, 1, \dots, 100\}$, and so $R_{\overline{X}} = \{0, \frac{1}{100}, \frac{2}{100}, \dots, 1\}$. Since $X_1, X_2, \dots$ are IIDRVs, we have

$$\overline{\mu} = \mu = \frac{1}{2}, \quad \overline{\sigma} = \frac{\sigma}{\sqrt{n}} = \sqrt{\frac{1}{2} \cdot \frac{1}{2}} \cdot \frac{1}{\sqrt{100}} = \frac{1}{20}$$

2. Explain why $P(|\overline{X} - \overline{\mu}| \leq \frac{1}{20})$ is same as the above summation. (3pts)

   *sol.* The inequality is equivalent to $-\frac{1}{20} \leq \overline{X} - \frac{1}{2} \leq \frac{1}{20} \Leftrightarrow 45 \leq X \leq 55$ where $X = X_1 + \cdots + X_{100}$. You can check that the random variable $X$ follows the binomial distribution with $n = 100$ and $p = \frac{1}{2}$, so

$$P(45 \leq X \leq 55) = \sum_{k=45}^{55} P(X = k) = \sum_{k=45}^{55} \binom{100}{k} \left(\frac{1}{2}\right)^k \left(1 - \frac{1}{2}\right)^{100-k} = \sum_{k=45}^{55} \binom{100}{k} \frac{1}{2^{100}}.$$

# Quiz 8



3. Using the given part of the standard normal table, approximate $P\left(|\overline{X} - \overline{\mu}| \leq \frac{1}{20}\right)$. (4pts)

*sol.* Consider the normalized random variable

$$Z = \frac{\overline{X} - \overline{\mu}}{\overline{\sigma}} = \frac{\overline{X} - 1/2}{1/20}$$

which satisfies $E[Z] = 0$ and $\mathrm{SE}[Z] = 1$. In terms of this random variable, the probability is same as

$$P(|\overline{X} - \overline{\mu}| \leq \frac{1}{20}) = P(|Z| \leq 1) = P(-1 \leq Z \leq 1),$$

and we can approximate this by standard normal distribution. In the table, you can see that the $z$-score for $z = 1$ is 0.3413, and this gives the value $P(0 \leq Z \leq 1)$. Hence $P(-1 \leq Z \leq 1) = 2 \cdot P(0 \leq Z \leq 1) = 0.6826$ since the standard normal distribution is symmetric.

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 |
|---|---|---|---|---|---|---|
| 0.0 | 0.0000 | 0.0040 | 0.0080 | 0.0120 | 0.0160 | 0.0190 |
| 0.1 | 0.0398 | 0.0438 | 0.0478 | 0.0517 | 0.0557 | 0.0596 |
| 0.2 | 0.0793 | 0.0832 | 0.0871 | 0.0910 | 0.0948 | 0.0987 |
| 0.3 | 0.1179 | 0.1217 | 0.1255 | 0.1293 | 0.1331 | 0.1368 |
| 0.4 | 0.1554 | 0.1591 | 0.1628 | 0.1664 | 0.1700 | 0.1736 |
| 0.5 | 0.1915 | 0.1950 | 0.1985 | 0.2019 | 0.2054 | 0.2088 |
| 0.6 | 0.2257 | 0.2291 | 0.2324 | 0.2357 | 0.2389 | 0.2422 |
| 0.7 | 0.2580 | 0.2611 | 0.2642 | 0.2673 | 0.2704 | 0.2734 |
| 0.8 | 0.2881 | 0.2910 | 0.2939 | 0.2969 | 0.2995 | 0.3023 |
| 0.9 | 0.3159 | 0.3186 | 0.3212 | 0.3238 | 0.3264 | 0.3289 |
| 1.0 | 0.3413 | 0.3438 | 0.3461 | 0.3485 | 0.3508 | 0.3513 |