Worksheet 3

More on the pigeonhole principle

1. **(original)** Show that in a section of 22 students, all of whom are first, second, or third years, at least one of the following must be true:
   (a) at least 15 are first years
   (b) at least 5 are second years
   (c) at least 4 are third years.

2. **(textbook 6.2.13)** Let (x_i, y_i, z_i), $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$ be a set of nine distinct points with integer coordinates in xyz-space. Show that the midpoints of at least one pair of these points has integer coordinates.

3. **(textbook 6.2.46)** There are 51 houses on a street. Each house has an address that is a positive integer between 1000 and 1099, inclusive. Show that at least two houses have addresses that are consecutive integers.

Combinations and Permutations

4. **(textbook 6.2.7)** Find the number of 5-permutations of a set with 9 elements.

5. **(textbook 6.2.17)** How many subsets with more than two elements does a set with 100 elements have?

6. **(textbook 6.2.23)** How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other?

7. **(textbook 6.2.37)** How many bit strings contain exactly eight 0s and ten 1s if every 0 must be immediately followed by a 1?

8. **(original)** A tennis competition has 6 players. In how many ways can three one-on-one matches be scheduled such that each player plays in exactly one match if
   (a) the three matches are distinguishable (for example, they happen at different times of the day)?
   (b) the three matches are indistinguishable (so that all that matters is who each player faces)?

9. **(classical, challenge problem)** Give a combinatorial proof of the fact that for $n \geq 1$, $0 \leq r \leq n-1$, $C(n, r) + C(n, r+1) = C(n+1, r+1)$.

Acknowledgments

All problems labeled "textbook" taken from Rosen, Kenneth H. Discrete Mathematics and its Applications. *Eighth edition. McGraw Hill, 2019.*