Worksheet 3

More on the pigeonhole principle

1. **(original)** Show that in a section of 22 students, all of whom are first, second, or third years, at least one of the following must be true:

   (a) at least 15 are first years

   (b) at least 5 are second years

   (c) at least 4 are third years.

   Suppose there are fewer than 15 first years. That means that there are at least 8 students who are eight second-years or third years. If there aren't at least 5 second years, then there are at most 4 second years and thus there must be at least 4 third years. So either there are 15 or more first years, or, if there aren't, there are either at least 5 second years or at least 4 third years. No matter what, one of the four options occurs.

   Aside from the "logical tree" approach above, another way to think about this problem is to see that if none of the scenarios above happen, there must be at most 14 first years, at most 4 second years, and at most 3 third years. Adding these up, we see that there are at most 21 students in the class, which can't happen because we are told that there are 22 students in the class.

2. **(textbook 6.2.13)** Let (x_i, y_i, z_i), $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$ be a set of nine distinct points with integer coordinates in xyz-space. Show that the midpoints of at least one pair of these points has integer coordinates.

   The midpoint of a pair of points (a, b, c) and (x, y, z) has integer coordinates only when all of $\frac{a+x}{2}$, $\frac{b+y}{2}$, and $\frac{c+z}{2}$ are integers. This in turn happens exactly when $a + x$, $b + y$, and $c + z$ are all even. This suggests that thinking about whether the coordinates of our points are even or odd will be useful.

   We know that the sum of two integers is even only when both integers are even or both integers are odd. We thus know that for the midpoint of two points with integer coordinates to be a point with integer coordinates, we need the two points to have their x coordinates the same parity (i.e. both even or both odd), their y coordinates the same parity, and their z coordinates the same parity.

   If we an ordered triple (x, y, z), there are 8 possible ways (do you see why?) for the elements to be even or odd. These ways will be the holes in our problem. Our pigeons will be the 9 given ordered triples. Since we have more pigeons than holes, there must be a hole with at least two pigeons in it. The midpoint of those two ordered triples will have all integer coordinates as desired.

3. **(textbook 6.2.46)** There are 51 houses on a street. Each house has an address that is a positive integer between 1000 and 1099, inclusive. Show that at least two houses have addresses that are consecutive integers.

We group addresses into sets of two, grouping 1000 and 1001, 1002 and 1003, and so on through 1098 and 1099. There are 50 such groups of two consecutive addresses. These will form our holes. The 51 houses will form our pigeons. Since we know all addresses are distinct, if we have two houses in the same group of two consecutive addresses, then the two addresses must be consecutive. But since we have more pigeons than holes, we know there is some hole with at least two pigeons in it, and thus two houses with consecutive addresses.

Combinations and Permutations

4. **(textbook 6.2.7)** Find the number of 5-permutations of a set with 9 elements.

   This is asking for $P(9,5)$; by our formula, we have $P(9,5) = \frac{9!}{5!} = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = \boxed{15120}$.

5. **(textbook 6.2.17)** How many subsets with more than two elements does a set with 100 elements have?

   The presence of "more than two" suggests complementary counting. We know there are 2^{100} total subsets of a set with 100 elements (this will be the size of our universe). The complement of the set we're trying to count is the set of subsets with less than or equal to two elements. This consists of the empty subset, the 100 one-element subset, and the $C(100,2) = 4950$ two-element subsets. We thus know that there are $2^{100} - 4950 - 100 - 1 = \boxed{2^{100} - 5051}$ subsets with more than two elements.

6. **(textbook 6.2.23)** How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other?

   We first order the eight men; there are $8! = 40320$ ways to do this. There are nine spaces where a woman could stand once we have ordered the men; in front of the line, or in the eight spaces immediately behind one of the men. Since we can't have two women standing next to each other, we need to choose exactly 5 of these 9 spaces to put a woman in; there are $C(9,5) = 126$ ways to do this. Once we have done so, we can order the 5 women however we want; there are $5! = 120$ ways to do this. The total number of arrangements of this form is thus

   $$8! \cdot C(9,5) \cdot 5! = 40320 \cdot 126 \cdot 120 = \boxed{609638400}.$$

7. **(textbook 6.2.37)** How many bit strings contain exactly eight 0s and ten 1s if every 0 must be immediately followed by a 1?

   Since every 0 must be immediately followed by a 1, we have eight blocks of 01 and then must put in two more 1s. There are 9 spaces we could put additional 1s: at the start of the string and after any 01 block. If both of the 1s are next to each other, we have 9 possible strings, one for each choice of where to put the 11 we are left with. If the 1s are not next to each other, then we need to choose two distinct spaces out of the 9; there are $C(9,2) = 36$ ways to do this. Adding up these two cases, we see that there are $\boxed{45}$ such bit strings.

8. **(original)** A tennis competition has 6 players. In how many ways can three one-on-one matches be scheduled such that each player plays in exactly one match if

(a) the three matches are distinguishable (for example, they happen at different times of the day)?

There are $C(6,2) = 15$ ways to pick the players who will face each other in the first match, as we need to pick two of the six players to face each other. There are $C(4,2) = 6$ ways to pick the players who will face each other in the second match, as there are only 4 players left after we have scheduled the first match. There is $C(2,2) = 1$ way to pick the players who will face each other in the third match, as it is completely determined by who we have picked in the first two matches. Multiplying these three numbers, we find $15 \cdot 6 \cdot 1 = \boxed{90}$ ways to schedule the matches.

(b) the three matches are indistinguishable (so that all that matters is who each player faces)?

When we went from the formula for permutations to the formula for combinations, we used the division rule to account for the fact that the formula $P(n,r)$ overcounts the number of subsets of size r since it counts the $r!$ ways to order the elements as distinct. Similarly, to solve this problem, we can take our answer from part (a) above and divide it by $3! = 6$, the number of ways to order the three matches (and thus the factor by which the answer in part (a) overcounts what we want in part (b)). Dividing 90 by 6, we see that there are $\boxed{15}$ ways to schedule the matches.

9. **(classical, challenge problem)** Give a combinatorial proof of the fact that for $n \geq 1$, $0 \leq r \leq n-1$, $C(n,r) + C(n,r+1) = C(n+1,r+1)$.

The expression $C(n+1,r+1)$ represents the number of ways to choose a subset of $r+1$ elements from a set of size $n+1$. Given a set of size $n+1$, we can pick some element e of the set and consider it. The number of ways to make a subset of size $r+1$ that contains e is $C(n,r)$, as we need to choose r more elements for our subset and we have n other elements to choose from. The number of ways to make a subset of size $r+1$ that does not contain e is $C(n,r+1)$, as we need to choose $r+1$ elements and must choose from the other n elements of our set. Since the element e has to either be in our subset or not in our subset, these two cases are the only possible cases, and, since they are disjoint, the number of ways to choose a subset of $r+1$ elements from a set of size $n+1$ is exactly $C(n,r) + C(n,r+1)$. Thus $C(n,r) + C(n,r+1) = C(n+1,r+1)$ as desired.

Acknowledgments

All problems labeled "textbook" taken from Rosen, Kenneth H. Discrete Mathematics and its Applications. *Eighth edition. McGraw Hill, 2019.*