# Worksheet 4



**More combinations and permutations**

1. How many permutations of the letters $ABCDEFG$ contain:

(a) the string $ACE$?

(b) the strings $AG$ and $FCB$?

(c) the strings $AB$, $DC$, and $GE$?

(d) the strings $ACB$ and $GFE$?

(a) We treat ACE as a single block and count permutations of $(ACE), B, D, F, G$. This is $P(5,5) = 5! = 120$.

(b) We treat $AG$ and $FCB$ as blocks and count permutations of $(AG), (FCB), D, E$. This is $P(4,4) = 4! = 24$.

(c) We treat $AB, DC$, and $GE$ as blocks and count permutations of $(AB), (DC), (EG)$, and $F$. This is again $P(4,4) = 4! = 24$.

(d) You know the drill. This is the number of permutations of $(ACB), (GFE)$, and $D$, which is $P(3,3) = 6$.

2. Ten women and eight men are on the faculty of a mathematics department at a school.

(a) How many ways are there to select a committee of five members of the department if at least one woman must be on the committee?

(b) How many ways are there to select a committee of five members of the department if at least one man and at least one woman must be on the committee?

(a) There are $\binom{18}{5}$ ways to choose a committee of five members from the eighteen faculty members. Of these $\binom{8}{5}$ will be committees consisting of all men. So $\binom{18}{5} - \binom{8}{5}$ is what we want.

(b) The previous part says $\binom{18}{5} - \binom{8}{5}$ count the number of committees with at least one woman on the committee. Of these, $\binom{10}{5}$ will correspond to committees with only women. So $\binom{18}{5} - \binom{8}{5} - \binom{10}{5}$ is what we want.

**Binomial coefficients**

3. What is the coefficient of $x^6y^{10}$ in the expansion of $(2x+5y)^{16}$?

This term in the expansion is $\binom{16}{6} \cdot (2x)^6 \cdot (5y)^{10} = 2^6 \cdot 5^{10} \cdot \binom{16}{6} \cdot x^6y^{10}$.

4. The row of Pascal's triangle containing the binomial coefficients $\binom{10}{k}, 0 \leq k \leq 10$, is:

1 10 45 120 210 252 210 120 45 10 1

Use Pascal's identity to produce the row immediately preceding this row in Pascal's triangle.

We start with a 1 up top and 10 down below, and subtract the last number we computed from the next number in the bottom sequence. (E.g. $10-1=9$ is next in the row we're computing, and then we do $45-9=36$ for the number afterward.) This gives us

1 9 36 84 126 126 84 36 9 1

5. Show that if $n$ and $k$ are integers with $1 \leq k \leq n$, then $\binom{n}{k} \leq n^k/2^{k-1}$.

Take the expansion

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+2)(n-k+1)}{k \cdot (k-1) \cdots 3 \cdot 2}.$$

Replacing each term in the numerator with the larger number $n$ tells us that

$$\binom{n}{k} \leq \frac{n^k}{k \cdot (k-1) \cdots 3 \cdot 2}.$$

Replacing each term in the denominator with the smaller number 2 then tells us that

$$\binom{n}{k} \leq \frac{n^k}{2^{k-1}}.$$

6. Prove the **hockeystick identity**:

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

whenever $n$ and $r$ are positive integers.

Look at Pascal's triangle. The terms on the left-hand side of the identity form a diagonal starting from the leftmost border of the triangle downward. Cascading applications of Pascal's identity will prove the result.

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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | 1 | | | | | | | | |
| | | | | | | | 1 | | 1 | | | | | | | |
| | | | | | | 1 | | 2 | | 1 | | | | | | |
| | | | | | 1 | | 3 | | 3 | | 1 | | | | | |
| | | | | 1 | | 4 | | 6 | | 4 | | 1 | | | | |
| | | | 1 | | 5 | | 10 | | 10 | | 5 | | 1 | | | |
| | | 1 | | 6 | | 15 | | 20 | | 15 | | 6 | | 1 | | |
| | 1 | | 7 | | 21 | | 35 | | 35 | | 21 | | 7 | | 1 | |
| 1 | | 8 | | 28 | | 56 | | 70 | | 56 | | 28 | | 8 | | 1 |

**More counting**

7. How many ways are there to distribute

(a) 10 distinguishable balls into four distinguishable bins?

(b) 10 indistinguishable balls into four distinguishable bins?

(a) We have four options for each ball, so the answer is $4^{10}$.

(b) We imagine the ten balls lined up, and placing three vertical dividers to split the balls into four groups. There are $\binom{10+4-1}{4-1} = \binom{13}{3}$ ways of doing this.

8. How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a piggy bank contain if it has 14 coins in it?

This is an indistinguishable balls and distinguishable bins question: the bins are the types of coins and the coins themselves are balls. So we have fourteen indistinguishable balls and five distinguishable bins, hence $\binom{14+5-1}{14} = \binom{18}{14}$ different combinations.

9. How many solutions are there to the inequality $x_1 + x_2 + x_3 + x_4 \leq 15$?

We add a fifth variable $x_5$ which picks up the slack of the rest of the sum, so that $x_1 + x_2 + x_3 + x_4 + x_5 = 15$ always. This is then the number of ways of fitting fifteen indistinguishable balls into five distinguishable urns, so $\binom{15+5-1}{15} = \binom{19}{15}$.

10. In the Chinese game Tractor, 48 of the 54 cards of a standard deck (counting jokers) are dealt to four players. How many different ways are there to deal Tractor hands to the four players?

We assume the four players are distinguishable. There are $\binom{54}{12}$ ways of dealing a hand to the first player, $\binom{42}{12}$ ways of then dealing a hand to the second player, $\binom{30}{12}$ ways of dealing a hand to the third player, and $\binom{18}{12}$ ways of dealing a hand to the fourth player. This gives

$$\binom{54}{12}\binom{42}{12}\binom{30}{12}\binom{18}{12} = \frac{54!}{42!12!} \cdot \frac{42!}{30!12!} \cdot \frac{30!}{18!12!} \cdot \frac{18!}{6!}12! = \frac{54!}{12!12!12!12!8!}.$$