Worksheet 5

more counting

1. How many ways are there to distribute

   i) 4 distinguishable balls into 10 distinguishable bins if no bin contains more than 1 ball?

   ii) 10 indistinguishable balls into 4 distinguishable bins if each bin contains at least 1 ball?

Solution:

i) $P(10,4) = \frac{10!}{6!}$ (distinguishable balls, distinguishable urns, injections)

ii) $\binom{10-1}{10-4} = \binom{9}{6}$ (indistinguishable balls, distinguishable urns, surjections)

2. How many permutations of MISSISSIPPI start with I?

Solution: A permutation of MISSISSIPPI that starts with I is the same as one I followed by a permutation of MISSISSIPP. Since there are $\frac{10!}{1!2!3!4!}$ permutations of MISSISSIPP, there are $\frac{10!}{1!2!3!4!}$ permutations of MISSISSIPPI that start with I.

3. (*) How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 20$ where x_1, x_2, x_3, x_4 are nonnegative integers where

   i) $x_1 \geq 4$?

   ii) $x_1 \leq 6$?

   iii) $2 \leq x_1 \leq 5$?

Solution:

i) Replace x_1 with $X_1 = x_1 - 4$. We now need to count the number of solutions to $X_1 + x_2 + x_3 + x_4 = 16$ where X_1, x_2, x_3, x_4 are nonnegative integers. This is the problem of putting 16 indistinguishable objects into 4 distinguishable boxes, so the answer is $\binom{4+16-1}{4-1} = \binom{19}{3}$.

ii) Similarly as in the previous part, there are $\binom{4+(20-7)-1}{4-1} = \binom{16}{3}$ solutions with $x_1 \geq 7$ and there are $\binom{4+20-1}{4-1} = \binom{23}{3}$ solutions overall. Thus there are $\binom{23}{3} - \binom{16}{3}$ solutions with $x_1 \leq 6$.

iii) Similarly as in the previous parts, there are $\binom{4+(20-6)-1}{4-1} = \binom{17}{3}$ solutions with $x_1 \geq 6$ and $\binom{4+(20-2)-1}{4-1} = \binom{21}{3}$ solutions with $x_1 \geq 2$. Thus there are $\binom{21}{3} - \binom{17}{3}$ solutions with $2 \leq x_1 \leq 5$.

4. How many ways are there to deal 8 cards from a deck of 52 cards to 3 players (allowing the possibility of a player getting zero cards) if

   i) the players are distinguishable?

   ii) the players are indistinguishable? (Your answer may be in terms of Stirling numbers.)

Solution:

i) There are $\binom{52}{8}$ ways to choose the 8 cards. After choosing the 8 cards, there are 3^8 ways to deal the 8 cards to the 3 players (cards = balls, players = urns). Thus there are $\binom{52}{8} \cdot 3^8$ ways.

ii) There are $\binom{52}{8}$ ways to choose the 8 cards. After choosing the 8 cards, there are $S(8,1) + S(8,2) + S(8,3)$ ways to deal the 8 cards to the 3 players (cards = balls, players = urns). Thus there are $\binom{52}{8} \cdot (S(8,1) + S(8,2) + S(8,3))$ ways.

5. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 21$ where the order of the integers x_1, x_2, x_3, x_4 does not matter and $x_1, x_2, x_3, x_4 \geq 4$?

Solution:
Replace each x_i with $X_i = x_i - 4$. We now need to count the number of solutions to $X_1 + X_2 + X_3 + X_4 = 5$ where the order of the integers x_1, x_2, x_3, x_4 does not matter and $x_1, x_2, x_3, x_4 \geq 0$. This is the problem of putting 5 indistinguishable objects into 4 indistinguishable boxes, so the answer is $p_1(5) + p_2(5) + p_3(5) + p_4(5)$ where $p_k(n)$ is the number of partitions of n into k positive integers.

$$\begin{aligned} p_1(5) &= 1 & & (5) \\ p_2(5) &= 2 & & (4+1, 3+2) \\ p_3(5) &= 2 & & (3+1+1, 2+2+1) \\ p_4(5) &= 1 & & (2+1+1+1) \end{aligned}$$

Therefore the answer is $p_1(5) + p_2(5) + p_3(5) + p_4(5) = 1 + 2 + 2 + 1 = 6$.