Worksheet 6

Algorithms

1. Demonstrate bubble sort to sort the list 3, 4, 2, 1.

Solution: On the first pass, we go 3421 → 3241 → 3214, on the next pass we do 3214 → 2314 → 2134 and on the third pass we do 2134 → 1234 and on the fourth pass, we make no changes which means the algorithm terminates.

2. Demonstrate the quick sort to sort the list $3, 6, 2, 5, 1, 4$.

Solution: First we place the first number in the appropriate position and put the numbers smaller before it and the numbers larger after while preserving their relative order to get 362514 → 213654. Now we do the same on the smaller numbers and the larger to get 21 → 12 and 654 → 546 → 456. This finally sorts the list as 123456.

3. Demonstrate the stable matching algorithm when men and women have the preferences $m_1 : w_1 > w_2, m_2 : w_1 > w_2$ and $w_1 : m_1 > m_2, w_2 : m_1 > m_2$.

Solution: Both men will propose to woman 1 and she will choose man number 1. Then man 2 will propose to his next option which is woman 2 and she will accept him. Thus, we get the final pairing $(m_1, w_1), (m_2, w_2)$.

4. Three women A, B, C are proposing to men E, F, G. Their preferences are as follows:

A	B	C	E	F	G
$E > G > F$	$E > G > F$	$G > E > F$	$C > A > B$	$A > B > C$	$B > C > A$

Show the stable matching algorithm with the women proposing to the men by clearly showing all rounds in a table.

Solution:

Men	Rd 1	Rd 2	Rd 3	Rd 4	Rd 5
E	A, ~~B~~	A	~~A~~, C	C	$\boxed{\text{C}}$
F					$\boxed{\text{A}}$
G	C	~~C~~, B	B	B, ~~A~~	$\boxed{\text{B}}$

5. Sort the list $2, 1, 6, 4, 5, 3$ using both bubble sort and quicksort.

Solution:
Using bubble sort, we get

$$216453 \to 124536 \to 124356 \to 123456$$

Using quicksort using the last number as a pivot, we get

$$216453 \to 213645 \to (12)3(456).$$

Inductions

6. Prove using mathematical induction that for all $n \geq 1$,

$$1 + 4 + 7 + \cdots + (3n-2) = \frac{n(3n-1)}{2}.$$

Solution: Basis step: for $n = 1$, $1 = \frac{1(3 \cdot 1 - 1)}{2}$.

Inductive step: suppose that the equation is true for n, so that

$$\sum_{k=1}^{n}(3k-2) = \frac{n(3n-1)}{2}.$$

Then

$$\begin{aligned}\sum_{k=1}^{n+1}(3k-2) = \sum_{k=1}^{n}(3k-2) + (3n+1) &= \frac{n(3n-1)}{2} + (3n+1) \\ &= \frac{3n^2 - n + 6n + 2}{2} = \frac{3n^2+5n+2}{2} = \frac{(n+1)(3n+2)}{2} \\ &= \frac{(n+1)(3(n+1)-2)}{2}\end{aligned}$$

so it is also true for $n+1$. Hence it is true for all n by mathematical induction.

7. Prove that

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$

Solution: Basis step: for $n = 1$, $\frac{1}{1\cdot 3} = \frac{1}{3}$.

Inductive step: suppose that the equation is true for n, so that

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$

Then

$$\begin{aligned}\sum_{k=1}^{n+1} \frac{1}{(2k-1)(2k+1)} &= \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2n+1)(2n+3)} \\ &= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} = \frac{n(2n+3)+1}{(2n+1)(2n+3)} \\ &= \frac{2n^2+3n+1}{(2n+1)(2n+3)} = \frac{(2n+1)(n+1)}{(2n+1)(2n+3)} \\ &= \frac{n+1}{2n+3}\end{aligned}$$

so it is also true for $n+1$. Hence it is true for all n by mathematical induction.

8. (*) Prove using mathematical induction that for all $n \geq 1$, $6^n - 1$ is divisible by 5.

Solution: Basis step: for $n = 1$, $6^1 - 1 = 5$ is divisible by 5.

Inductive step: suppose that $6^n - 1$ is divisible by 5 for n. Then

$$6^{n+1} - 1 = 6(6^n - 1) + 6 - 1 = 6(6^n - 1) + 5.$$

Since both $6^n - 1$ and 5 are multiple of 5, so is $6^{n+1} - 1$. Hence it is true for all n by mathematical induction.

9. Let $\{a_n\}_{n\geq 1}$ be a sequence defined as $a_1 = 1$ and $a_{n+1} = \sqrt{a_n + 2}$. Prove that $a_n \leq 2$ for all $n \geq 1$, by using mathematical induction.

Solution: Basis step: for $n = 1$, $a_1 = 1 \leq 2$.

Inductive step: suppose that $a_n \leq 2$ for n. Then

$$a_{n+1} = \sqrt{a_n + 2} \leq \sqrt{2+2} = 2,$$

so it is also true for $n+1$. Hence it is true for all n by mathematical induction.