Worksheet 7

Inductions

1. Prove using mathematical induction that for all $n \geq 1$,

$$1 + 4 + 7 + \cdots + (3n-2) = \frac{n(3n-1)}{2}.$$

Solution: Basis step: for $n = 1$, $1 = \frac{1(3\cdot 1-1)}{2}$.

Inductive step: suppose that the equation is true for n, so that

$$\sum_{k=1}^{n}(3k-2) = \frac{n(3n-1)}{2}.$$

Then

$$\begin{aligned}\sum_{k=1}^{n+1}(3k-2) = \sum_{k=1}^{n}(3k-2) + (3n+1) &= \frac{n(3n-1)}{2} + (3n+1) \\ &= \frac{3n^2 - n + 6n + 2}{2} = \frac{3n^2+5n+2}{2} = \frac{(n+1)(3n+2)}{2} \\ &= \frac{(n+1)(3(n+1)-2)}{2}\end{aligned}$$

so it is also true for $n+1$. Hence it is true for all n by mathematical induction.

2. Prove that

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$

Solution: Basis step: for $n = 1$, $\frac{1}{1\cdot 3} = \frac{1}{3}$.

Inductive step: suppose that the equation is true for n, so that

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$

Then

$$\begin{aligned}\sum_{k=1}^{n+1} \frac{1}{(2k-1)(2k+1)} &= \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2n+1)(2n+3)} \\ &= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} = \frac{n(2n+3)+1}{(2n+1)(2n+3)} \\ &= \frac{2n^2+3n+1}{(2n+1)(2n+3)} = \frac{(2n+1)(n+1)}{(2n+1)(2n+3)} \\ &= \frac{n+1}{2n+3}\end{aligned}$$

so it is also true for $n + 1$. Hence it is true for all n by mathematical induction.

3. (*) Prove using mathematical induction that for all $n \geq 1$, $6^n - 1$ is divisible by 5.

Solution: Basis step: for $n = 1$, $6^1 - 1 = 5$ is divisible by 5.

Inductive step: suppose that $6^n - 1$ is divisible by 5 for n. Then

$$6^{n+1} - 1 = 6(6^n - 1) + 6 - 1 = 6(6^n - 1) + 5.$$

Since both $6^n - 1$ and 5 are multiple of 5, so is $6^{n+1} - 1$. Hence it is true for all n by mathematical induction.

4. Let $\{a_n\}_{n\geq 1}$ be a sequence defined as $a_1 = 1$ and $a_{n+1} = \sqrt{a_n + 2}$. Prove that $a_n \leq 2$ for all $n \geq 1$, by using mathematical induction.

Solution: Basis step: for $n = 1$, $a_1 = 1 \leq 2$.

Inductive step: suppose that $a_n \leq 2$ for n. Then

$$a_{n+1} = \sqrt{a_n + 2} \leq \sqrt{2+2} = 2,$$

so it is also true for $n + 1$. Hence it is true for all n by mathematical induction.

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5. Let $\{a_n\}_{n\geq 1}$ be a sequence defined as $a_1 = 1, a_2 = 5$ and $a_{n+2} = 5a_{n+1} - 6a_n$. Prove that $a_n = 3^n - 2^n$ for all $n \geq 1$, by using mathematical induction.

Solution: Basis step: for $n = 1$, $a_1 = 1 = 3^1 - 2^1$ and for $n = 2$, $a_2 = 5 = 3^2 - 2^2$.

Inductive step: suppose that the statement holds for n and $n+1$. For $n+2$, we have

$$\begin{aligned} a_{n+2} &= 5a_{n+1} - 6a_n = 5(3^{n+1} - 2^{n+1}) - 6(3^n - 2^n) \\ &= 15 \cdot 3^n - 10 \cdot 2^n - 6 \cdot 3^n + 6 \cdot 2^n = 9 \cdot 3^n - 4 \cdot 2^n \\ &= 3^{n+2} - 2^{n+2}, \end{aligned}$$

so it is also true for $n+2$. Hence it is true for all n by mathematical induction.

6. (a) Prove that $n^2 + 3n$ can be divided by 2 for every $n \geq 1$.
   (b) Prove that $n^3 - n$ can be divided by 3 for every $n \geq 1$.

Solution: (a) Basis step: for $n = 1$, $1^2 + 3 \cdot 1 = 4$ can be divided by 2.

Inductive step: suppose that the statement holds for n. For $n+1$, we have

$$(n+1)^2 + 3(n+1) = n^2 + 2n + 1 + 3n + 3 = (n^2 + 3n) + 2n + 4 = (n^2 + 3n) + 2(n+2).$$

Since $n^2 + 3n$ can be divided by 2 by the assumption for n and $2(n+2)$ also can be divided by 2, their sum can be divided by 2. So it is also true for $n+1$. Hence it is true for all n by mathematical induction.

(b) Basis step: for $n = 1$, $1^3 - 1 = 0$ can be divided by 3.

Inductive step: suppose that the statement holds for n. For $n+1$, we have

$$(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - (n+1) = (n^3 - n) + 3(n^2 + n).$$

Since $n^3 - n$ can be divided by 3 by the assumption for n and $3(n^2 + n)$ also can be divided by 3, their sum can be divided by 3. So it is also true for $n+1$. Hence it is true for all n by mathematical induction.

7. (a) (*) Let a_n be the number of permutation of distinguishable n-balls. (Assume that we don't know $a_n = n!$ yet.) Prove that $a_1 = 1$ and $a_{n+1} = (n+1)a_n$.
   (b) By using the above recurrence relation and mathematical induction, prove that $a_n = n!$.

Solution: (a) Assume that we have a permutation of n distinct balls. If we try to put one more ball to make it as $(n+1)$ balls, there are $(n+1)$ choices to put the $(n+1)$-th ball: between these n-balls, left end or the right end. Since putting $(n+1)$-th ball is independent of the previous n-balls, we have $a_{n+1} = (n+1) \times a_n$. $a_1 = 1$ is trivial.

(b) Basis step: for $n = 1$, we have $a_1 = 1 = 1!$.

Inductive step: suppose that the statement holds for n, so that $a_n = n!$. For $n+1$, we have

$$a_{n+1} = (n+1)a_n = (n+1) \cdot n! = (n+1)!,$$

so it is also true for $n+1$. Hence it is true for all n by mathematical induction.