Basic Discrete Probability

1. What is the probability that a five-card poker hand contains a royal flush, that is, the 10, jack, queen, king, and ace of one suit?

Solution Here we can just enumerate possibilities by hand. There are four five-card hands that are royal flushes, one for each suit. The sample space has size $\binom{52}{5}$. Thus the probability is $\frac{4}{\binom{52}{5}}$.

2. What is the probability that Abby, Barry, and Sylvia win the first, second, and third prizes, respectively, in a drawing if 200 people enter a contest and

   (a) no one can win more than one prize.

Solution The probability is simply $\frac{1}{200} \cdot \frac{1}{199} \cdot \frac{1}{198}$, as we draw without replace independently.

   (b) winning more than one prize is allowed.

Solution This time it is $\frac{1}{200}^3$, since we draw with replacement.

3. What is the probability of these events when we randomly select a permutation of $\{1, 2, 3\}$?

   (a) 1 precedes 3.

Solution There are two permutations in which 1 precedes 3, 123, 213 and 132. Thus it is $\frac{3}{3!} = \frac{1}{2}$.

   (b) 3 precedes 1.

Solution Same as above.

   (c) 3 precedes 1 and 3 precedes 2.

Solution Here there are two permutations as well 312 and 321, giving $\frac{2}{3!} = \frac{1}{3}$.

4. Assume that the probability a child is a boy is 0.51 and that sexes of children born into a family are independent. What is the probability a family of five children has

   (a) exactly three boys?

Solution This is $\binom{5}{3} \cdot 0.51^3 \cdot 0.49^2$.

   (b) at least one boy?

Solution Here we can consider the complement (having all girls), yielding $1 - 0.49^5$.

   (c) at least one girl?

Solution Again we consider the complement to get $1 - 0.51^5$.

   (d) two boys, conditional on there being at least two girls?

Solution The probability of at least two girls is $1 - 0.51^5 - \binom{5}{4} \cdot 0.49 \cdot 0.51^4$. For there two be exactly two boys and at least two girls forces there to be two boys and three girls. This event has probability $\binom{5}{2} \cdot 0.49^3 \cdot 0.51^2$. Thus the probability is

$$\frac{\binom{5}{2} \cdot 0.49^3 \cdot 0.51^2}{1 - 0.51^5 - \binom{5}{4} \cdot 0.49 \cdot 0.51^4}$$

Compare with the unconditional probability $\binom{5}{2} \cdot 0.49^3 \cdot 0.51^2$.

5. Assume that the probability of a 0 is 0.8 and a 1 is 0.2 for a randomly generated bit string of length six. What is the probability that there

(a) are at least 3 zeros?

Solution Here it is $\binom{6}{3}\cdot 0.8^3\cdot 0.2^3+\binom{6}{4}\cdot 0.8^4\cdot 0.2^2+\binom{6}{5}\cdot 0.8^5\cdot 0.2+0.8^6$. Notice that the complement is not much less work.

(b) are two ones, conditional on the first digit being a zero?

Solution The first digit is a zero with probability 0.8. The intersection is the probability the first digit is a zero and there are two ones, which is $\binom{5}{2}\cdot 0.8^4\cdot 0.2^2$. Thus the probability is $\frac{\binom{5}{2}\cdot 0.8^4\cdot 0.2^2}{0.8}=\binom{5}{2}\cdot 0.8^3\cdot 0.2^2$.

(c) is a run of exactly two zeros in a row?

Solution We will count the number of strings with fixed number of zeros in it, satisfying the condition.

- 2 zeros: Since two zeros should be adjacent, we have 5 strings: 001111, 100111, ..., 111100.
- 3 zeros: Two of zeros should be adjacent and the other one should be separated from that pair. If the string starts with 00, then third bit should be 1, so is 001***, and we have 3 choices for ***. This is same for the string ends with two zeros. If the pair is not on the start or end of the string (such as ***00*), then the bits next to these zeros should be 1 (**1001) and we have 2 choices for the rest 2 bits. There are 3 such strings. Hence we have $2\times 3+3\times 2=12$ strings with 3 zeros where two of them are adjacent.
  There is an easier way to do this. This is a variant of the problem about marking books on a shelf, where the markings are not adjacent. We will arrange a block 00 and 0 and 3 one's, where 00 block and 0 are not adjacent. To do this, we put ones first, and we have 4 places to put 00 block and 0:
  $$()1()1()1()$$
  since 00 block and 0 are different, order matters and the answer is $P(4,2)=4\times 3=12$.
- 4 zeros: There are 6 such strings among $\binom{6}{4}=15$ strings with 4 zeros: 100100, 010100, 010010, 001100, 001010, 001001.
- 5 zeros: There are 2 such strings among $\binom{6}{5}=6$ strings with 5 zeros: 001000, 000100.

Now, for fixed number of zeros, the probability for the string is same. If there are r many zeros in the string, then the probability of the string to occur is $0.8^r 0.2^{6-r}$. Hence the answer is

$$5\cdot 0.8^2\cdot 0.2^4+12\cdot 0.8^3\cdot 0.2^3+6\cdot 0.8^4\cdot 0.2^2+2\cdot 0.8\cdot 0.2^5$$

(If the problem only considers string with *exactly one* pair of adjacent zeros, then the answer is $5\cdot 0.8^2\cdot 0.2^4$.)

(d) is a run of exactly two zeros in a row, conditional on the last digit being a one?

Solution The probability that last digit is one is 0.2. If we count strings with a pair of adjacent zeros *and* ends with one, then there are

- 4 with 2 zeros
- 6 with 3 zeros: We don't have to care about the last digit (which is one) and we will arrange a block 00, 0, and two ones. As we did in (c), we put ones first and we have 3 places to put the 00 block and single 0:

$$()1()1()$$

  and we get $P(3,2) = 6$.
- 1 with 4 zeros

(and no with 5 zeros) so the probability for the intersection is $4 \cdot 0.8^2 \cdot 0.2^4 + 6 \cdot 0.8^3 \cdot 0.2^3 + 0.8^4 \cdot 0.2^2$. Hence the answer is

$$\frac{4 \cdot 0.8^2 \cdot 0.2^4 + 6 \cdot 0.8^3 \cdot 0.2^3 + 0.8^4 \cdot 0.2^2}{0.2}$$

(If the problem only considers string with *exactly one* pair of adjacent zeros, then the answer is $\frac{4 \cdot 0.8^2 \cdot 0.2^4}{0.2} = 4 \cdot 0.8^2 \cdot 0.2^3$.)

Source: Rosen's *Discrete Mathematics and its Applications*.