Worksheet 11

1. Concept of Independence

   - What is the probability that two people chosen at random were born during the same month of the year?

     Solution: $\binom{12}{1}\frac{1}{12}\frac{1}{12} = \frac{1}{12}$

   - What is the probability that in a group of n people chosen at random, there are at least two born in the same month of the year?

     Solution: If $n > 12$, by the pigeonhole principle, at least two people were born in the same month. Otherwise, $P(\text{at least two born in the same month}) = 1 - P(\text{all born in different month}) = 1 - \frac{12}{12} \times \frac{11}{12} \times \frac{10}{12} \dots \frac{12-n+1}{12}$.

   - How many people chosen at random are needed to make the probability greater than 1/2 that there are at least two people born in the same month of the year?

     Solution: In other words, we want to find the minimum n such that $1 - P(\text{all born in different month}) = 1 - \frac{12}{12} \times \frac{11}{12} \times \frac{10}{12} \dots \frac{12-n+1}{12} > \frac{1}{2}$. When $n = 4, P(\text{at least two born in the same month}) \approx 0.482$, and When $n = 5, P(\text{at least two born in the same month}) \approx 0.618$. Thus, $n \geq 5$.

2. Two dies were rolled. Are the events that the first die rolled is a 1 and that the sum of the two dice is a 7 independent?

   Solution: Let A be the event that first die rolled is a 1, and B be the event that the sum of the two dice is a 7. Then,

   $$P(A \cap B) = \frac{1}{36} \quad (1)$$
   $$P(A) = \frac{1}{6} \quad (2)$$
   $$P(B) = \frac{6}{36} \quad (3)$$

   Therefore, $P(A \cap B) = P(A)P(B)$.

3. Find the smallest number of people you need to choose at random so that the probability that at least two of them were both born on April 1st exceeds 1/2.

> **Solution:** P(at least two of them were both born on April 1st) =
> $1 - P$(none of them were both born on April 1st)
> $-P$(exactly one of them were both born on April 1st) $= 1-(\frac{365}{366})^n - \binom{n}{1}(\frac{1}{366})(\frac{365}{366})^{(n-1)}$
> $> \frac{1}{2}$. Solving the equation, we have $n > 614$.

4. Assume that the probability a child is a boy is 0.51 and that the sexes of children born into a family are independent. What is the probability that a family of five children has

   - exactly three boys?

     > **Solution:** $\binom{5}{3}(0.51)^3(0.49)^2$

   - at least one boy?

     > **Solution:** $1 - 0.49^5$

   - at least one girl?

     > **Solution:** $1 - 0.51^5$

   - all children of the same sex?

     > **Solution:** $0.51^5 + 0.49^5$

5. Let E be the event that a randomly generated bit string of length three contains an odd number of 1s, and let F be the event that the string starts with 1. Are E and F independent?

> **Solution:**
> $$P(E) = \frac{4}{8}$$
> $$P(F) = \frac{1}{2}$$
> $$P(E \cap F) = \frac{2}{8} = P(E)P(F)$$

Therefore, independent.

6. Let E and F be the events that a family of n children has children of both sexes and has at most one boy, respectively. Are E and F independent if

   - $n = 2$
   - $n = 4$
   - $n = 5$

Solution:

$$P(E) = 1 - 2 \times (0.5)^n$$

$$P(F) = (0.5)^n + \binom{n}{1}(0.5)^n = (n+1)(0.5)^n$$

$$P(E \cap F) = \binom{n}{1}(0.5)^n$$

When $n = 2, P(E \cap F) = 0.5$ and $P(E) \times P(F) = 0.375$;
When $n = 4, P(E \cap F) = 0.25$ and $P(E) \times P(F) = 0.2734375$;
When $n = 5, P(E \cap F) = 0.15625$ and $P(E) \times P(F) = 0.17578125$;
Therefore, E and F are not independent for $n = 2, 4$ or 5.

Source: Rosen's *Discrete Mathematics and its Applications.*