Worksheet 17

PMF, PDF, CDF

1. Find a value c such that the given function became PDF. If there's no such c, explain why. Also, find corresponding CDFs and compute $P(0 \le X \le 2)$.

   (a) $f(x) = \begin{cases} c & 0 \le x \le 25 \\ 0 & \text{otherwise} \end{cases}$

   sol. This is a uniform distribution, and $\int_{-\infty}^{\infty} f(x)dx = \int_0^{25} c dx = 25c = 1$, so $c = 1/25$. The corresponding CDF is

$$F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0 & x \le 0 \\ \frac{1}{25}x & 0 < x \le 25 \\ 1 & x > 25 \end{cases}$$

   The probability is $P(0 \le X \le 2) = \int_0^2 f(x)dx = F(2) - F(0) = \frac{2}{25}$.

   (b) $f(x) = \begin{cases} cx(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

   sol. $\int_{-\infty}^{\infty} f(x)dx = \int_0^1 c(x - x^2)dx = c[\frac{1}{2}x^2 - \frac{1}{3}x^3]_0^1 = \frac{c}{6} = 1$, so $c = 6$. The corresponding CDF is

$$F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0 & x \le 0 \\ 3x^2 - 2x^3 & 0 < x \le 1 \\ 1 & x > 1 \end{cases}$$

   and $P(0 \le X \le 2) = \int_0^2 f(x)dx = F(2) - F(0) = 1$.

   (c) $f(x) = \begin{cases} c(x^2 - 1) & -2 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$

   sol. If this is a PDF of some continuous random variable, then $\int_{-\infty}^{\infty} f(x)dx = \int_{-2}^{2} c(x^2 - 1)dx = c[\frac{x^3}{3} - x]_{-2}^{2} = \frac{4c}{3} = 1$, so $c = \frac{3}{4}$. However, for such c, $f(x)$ is negative when $-1 < x < 1$. Hence $f(x)$ can't be a PDF.

   (d) $f(x) = ce^{-|x|}$

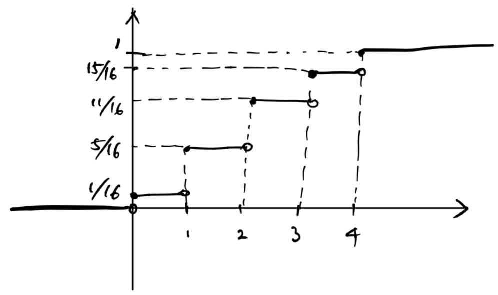
   sol. $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} ce^{-|x|}dx = 2c\int_0^{\infty} e^{-x}dx = 2c = 1$, so $c = \frac{1}{2}$. The corresponding CDF is

$$F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} \frac{1}{2}e^{x} & x < 0 \\ 1 - \frac{1}{2}e^{-x} & x \ge 0 \end{cases}$$

   and $P(0 \le X \le 2) = \int_0^2 f(x)dx = F(2) - F(0) = \frac{1}{2} - \frac{1}{2}e^{-2}$.

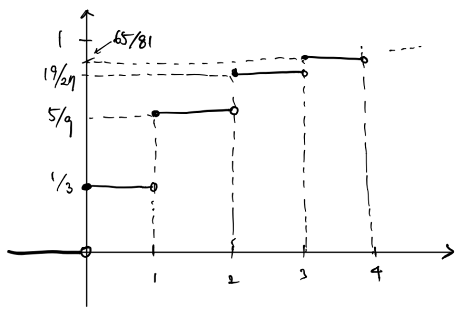
2. Find CDF of a binomial distribution with $n = 4, p = 1/2$.

   sol.



3. (*) Find CDF of a geometric distribution with $p = 1/3$.

sol. (This graph keeps increasing but never touches 1.)



4. (a) Suppose that the probability density function P that an atom emits a gamma wave with the PDF $f(t) = Cte^{-t^2}$ for $t \geq 0$ and $f(t) = 0$ for $t < 0$. Find $f(t)$ and calculate the CDF of $f(t)$.

sol. Use substitution $u = t^2, du = 2tdt$. Then

$$\int_{-\infty}^{\infty} f(t)dt = \int_{0}^{\infty} Cte^{-t^2} dt = \frac{C}{2}\int_{0}^{\infty} e^{-u}du = \frac{C}{2}[-e^{-u}]_0^{\infty} = \frac{C}{2} = 1$$

so $C = 2$ and $f(t) = 2te^{-t^2}$. CDF is

$$F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-x^2} & x > 0 \end{cases}$$

(b) For the above PDF, find the probability that a gamma wave is emitted from -2 seconds to 2 seconds.

sol. $P(-2 \leq X \leq 2) = \int_{-2}^{2} f(x)dx = F(2) - F(-2) = 1 - e^{-4}$.

Worksheet 17

5. For given CDF, compute the probability $P(-1 \leq X \leq 1)$ and find corresponding PDF (or PMF).

   (a) $F(x) = \begin{cases} 0 & x \leq -2 \\ \frac{1}{4}x + \frac{1}{2} & -2 < x < 2 \\ 1 & x \geq 2 \end{cases}$

   sol. $P(-1 \leq X \leq 1) = F(1) - F(-1) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$. The corresponding PDF is a uniform distribution, given by

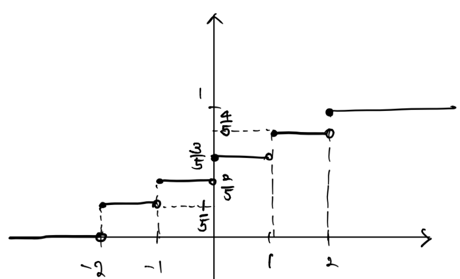
   $$f(x) = \begin{cases} \frac{1}{4} & -2 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

   (b) (*) $F(x) = A \arctan x + B$ (find A and B.)

   sol. First, if $F(x)$ is a CDF, we should have $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$. Since $\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}$ and $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$, we have $-\frac{\pi}{2}A + B = 0$ and $\frac{\pi}{2}A + B = 1$. This implies $B = \frac{1}{2}$ and $A = \frac{1}{\pi}$, so $F(x) = \frac{1}{\pi}\arctan x + \frac{1}{2}$. This gives $P(-1 \leq X \leq 1) = F(1) - F(-1) = \frac{1}{\pi}(\arctan 1 - \arctan(-1)) = \frac{1}{\pi}(\frac{\pi}{4} - (-\frac{\pi}{4})) = \frac{1}{2}$. The corresponding PDF can be obtained by differentiating $F(x)$, so that

   $$f(x) = F'(x) = \frac{1}{\pi(1+x^2)}.$$

   (c) (*)



   Since CDF is a *step function* (looks like stairs), it is a CDF of a discrete random variable. We should be careful in this case, because $P(-1 \leq X \leq 1)$ is NOT $F(1) = F(-1) = \frac{2}{5}$. PMF of corresponding random variable is $f(k) = P(X = k) = \frac{1}{5}$ for each $k \in \{-2, -1, 0, 1, 2\}$, so $P(-1 \leq X \leq 1) = f(-1) + f(0) + f(1) = \frac{3}{5}$.