Worksheet 25

Chi-Squared Hypothesis Testing

1. You roll a die 60 times and get 10 1's, 10 2's, 10 3's, 2 4's, 8 5's, and 20 6's. Is the die fair? Use the significance level $\alpha = 0.05$.

   (a) What are H_0 and H_1?

   Solution:
   H_0: The die is fair.
   H_1: The die is not fair.

   (b) Complete the following table.

value k	observed frequency n_k	expected frequency m_k	$(n_k - m_k)^2/m_k$
1	10		
2	10		
3	10		
4	2		
5	8		
6	20		

   Solution:

value k	observed frequency n_k	expected frequency m_k	$(n_k - m_k)^2/m_k$
1	10	10	0
2	10	10	0
3	10	10	0
4	2	10	6.4
5	8	10	0.4
6	20	10	10

   (c) Calculate the χ^2 statistic and determine the number of degrees of freedom.

   Solution: The χ^2 statistic is $r = 0 + 0 + 0 + 6.4 + 0.4 + 10 = 16.8$. We have $6 - 1 = 5$ degrees of freedom.

   (d) Draw a conclusion.

   Solution:
   There are two possible methods.
   METHOD 1 - Using the χ^2 table, the critical χ^2 value is 11.07. Thus, since $16.8 > 11.07$, we reject H_0 in favor of H_1 i.e. we conclude that the die is not fair.
   METHOD 2 - Using the χ^2 calculator, $P(R \geq 16.8) \approx 0.01$. Thus, since $0.01 < 0.05$, again we reject H_0 in favor of H_1 i.e. we conclude that the die is not fair.

2. [HW34#1] We roll two 6-sided dice 100 times and record the outcomes for the sum of the dice in the following table.

value	observed frequency	expected frequency
2	6	
3	10	
4	9	
5	13	
6	13	
7	12	
8	11	
9	10	
10	7	
11	5	
12	4	

Calculate the expected frequencies, given the null hypothesis H_0 that both dice are fair. Compute the χ^2 statistic for this data. What is the p-value? Do we have enough evidence to reject the null hypothesis?

Solution:

value	observed frequency	expected frequency
2	6	$\frac{1}{36} \cdot 100$
3	10	$\frac{2}{36} \cdot 100$
4	9	$\frac{3}{36} \cdot 100$
5	13	$\frac{4}{36} \cdot 100$
6	13	$\frac{5}{36} \cdot 100$
7	12	$\frac{6}{36} \cdot 100$
8	11	$\frac{5}{36} \cdot 100$
9	10	$\frac{4}{36} \cdot 100$
10	7	$\frac{3}{36} \cdot 100$
11	5	$\frac{2}{36} \cdot 100$
12	4	$\frac{1}{36} \cdot 100$

$$r = \frac{(6 - \frac{1}{36} \cdot 100)^2}{\frac{1}{36} \cdot 100} + ... + \frac{(4 - \frac{1}{36} \cdot 100)^2}{\frac{4}{36} \cdot 100} = 10.55$$

Using the χ^2 calculator, since we have $11 - 1 = 10$ degrees of freedom, the p-value is 0.3936. Since $0.3936 > 0.05$, we do not have enough evidence to reject H_0.

3. I claim that a coin is biased so that the probability of heads is 75%. When you flip the coin 40 times, you get 25 heads and 15 tails. Do you have enough evidence to reject my claim? Use the significance level $\alpha = 0.05$.

Solution:
Let H_0 be the hypothesis that my claim is true i.e. the probability of heads is 75%. Let H_1 be the hypothesis that my claim is false. Assuming H_0, the expected number of heads is $0.75 \cdot 40 = 30$ and the expected number of tails is $(1 - 0.75) \cdot 40 = 10$. Thus

$$r = \frac{(25-30)^2}{30} + \frac{(15-10)^2}{10} = \frac{10}{3}.$$

Using the χ^2 table, since $\alpha = 0.05$ and we have $2 - 1 = 1$ degree of freedom, the critical χ^2 value is 3.84. Thus, since $\frac{1}{3} < 3.84$, we fail to reject H_0 i.e. you do not have enough evidence to reject my claim.

4. In a sample of 160 pea plants, we observe 100 tall purple plants, 23 tall white plants, 25 short purple plants, and 12 short white plants. Let the null hypothesis H_0 be that flower color and plant height are Mendelian traits. Let the alternative hypothesis H_1 be that flower color and plant height are not Mendelian traits. Using the significance level $\alpha = 0.05$, do we have enough evidence to reject H_0? (Recall that we expect the proportion of the four possible phenotypes (TP, TW, SP, SW) to be 9:3:3:1 if flower color and plant height are Mendelian.)

Solution:

value	observed frequency	expected frequency
TP	100	$\frac{9}{16} \cdot 160 = 90$
TW	23	$\frac{3}{16} \cdot 160 = 30$
SP	25	$\frac{3}{16} \cdot 160 = 30$
SW	12	$\frac{1}{16} \cdot 160 = 10$

$$r = \frac{(100-90)^2}{90} + \frac{(23-30)^2}{30} + \frac{(25-30)^2}{30} + \frac{(12-10)^2}{10} \approx 3.98$$

Using the χ^2 table, since $\alpha = 0.05$ and we have $4 - 1 = 3$ degrees of freedom, the critical χ^2 value is 7.81. Thus, since $3.98 < 7.81$, we do not have enough evidence to reject H_0.