Worksheet 27

Partial derivatives & Differential equations

1. Compute partial derivatives $f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}$ of given functions, and check that $f_{xy} = f_{yx}$ holds.

   (a) $f(x, y) = e^{xy}$

   (b) $f(x, y) = \sin(2x + y)$

   (c) $f(x, y) = x \ln y + y \ln x$

   (d) $f(x, y) = x^y$

   sol.

   (a) $f_x = ye^{xy}, f_y = xe^{xy}, f_{xx} = y^2 e^{xy}, f_{xy} = (1 + xy)e^{xy}, f_{yx} = (1 + xy)e^{xy}, f_{yy} = x^2 e^{xy}$.

   (b) $f_x = 2\cos(2x + y), f_y = \cos(2x + y), f_{xx} = -4\sin(2x + y), f_{xy} = -2\sin(2x + y), f_{yx} = -2\sin(2x + y), f_{yy} = -\sin(2x + y)$.

   (c) $f_x = \ln y + \frac{y}{x}, f_y = \frac{x}{y} + \ln x, f_{xx} = -\frac{y}{x^2}, f_{xy} = \frac{1}{y} + \frac{1}{x}, f_{yx} = \frac{1}{x} + \frac{1}{y}, f_{yy} = -\frac{x}{y^2}$

   (d) $f_x = yx^{y-1}, f_y = x^y \ln x, f_{xx} = y(y-1)x^{y-2}, f_{xy} = x^{y-1} + yx^{y-1}\ln x, f_{yx} = yx^{y-1}\ln x + x^{y-1}, f_{yy} = x^y(\ln x)^2$

2. Let $z = \sin(x^2 + y)$ and $x = e^t, y = \frac{1}{t}$. Compute $\frac{dz}{dt}$.
   sol. By the chain rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = 2x\cos(x^2 + y)e^t + \cos(x^2 + y)\left(-\frac{1}{t^2}\right) = \cos\left(e^{2t} + \frac{1}{t}\right)\left(2e^{2t} - \frac{1}{t^2}\right)$$

3. (Wave equation) Let v be a fixed constant. The following partial differential equation

$$\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

   is called Wave equation. Prove that $y(x, t) = \sin(x - vt) + \sin(x + vt)$ is a solution of the Wave equation.
   sol. We have

$$\begin{aligned} y_t &= -v\cos(x - vt) + v\cos(x + vt), \quad y_{tt} = -v^2\sin(x - vt) - v^2\sin(x + vt) \\ y_x &= \cos(x - vt) + \cos(x + vt), \quad y_{xx} = -\sin(x - vt) - \sin(x + vt) \end{aligned}$$

   so we get $\frac{1}{v^2}y_{tt} = y_{xx}$.

Some review

1. How many ways can you rearrange the letters in BERKELEY?
   sol. We have 8! ways to rearrange them, and rearranging E's do not change the word, so the answer will be

$$\frac{8!}{3!} = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$$

2. How many ways can we split 9 people into 3 distinguishable teams with 3 people each? What if the teams are indistinguishable?
sol. (1) Distinguishable teams. Let's say, team A, B, and C. We can choose 3 for team A, which has $\binom{9}{3}$ choice. Then we can choose 3 for team B, which has $\binom{6}{3}$ choice. The remaining people will automatically be team C. So we have
$$\binom{9}{3}\binom{6}{3}\binom{3}{3} = \frac{9!}{6!3!}\frac{6!}{3!3!} = \frac{9!}{3!3!3!}.$$
If the teams are indistinguishable, rearranging teams will give the same thing. So we have to divide the above answer by 3! and we get
$$\frac{9!}{3!3!3!3!}.$$

3. What is the probability that a 5-card poker hand contains at least one ace?
sol. By the complement rule,
$$P(\text{at least one ace}) = 1 - P(\text{no aces}).$$
Among $\binom{52}{5}$ possible outcomes, $\binom{48}{5}$ do not contain any aces (choose 5 cards from 48 non-aces), so the answer will be
$$1 - \frac{\binom{48}{5}}{\binom{52}{5}} = 1 - \frac{47 \cdot 46 \cdot 45 \cdot 44}{52 \cdot 51 \cdot 50 \cdot 49}$$

4. The average number of chocolates sold by Trader's Joe is 150 chocolates per day. What is the probability that exactly 150 chocolates will be sold tomorrow?
sol. The number of chocolates X follows the Poisson distribution. Since $E[X] = \lambda$, we have $\lambda = 150$ and the probability is
$$P(X = 150) = \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-150}150^{150}}{150!}.$$