MATH 53 EXAM 2

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For full credit, please show all your work and reasoning. Your work and explanations are your only representative when your work is being graded. Illegible, messy, and mysterious work could be perceived as an error that undermines the grader's ability to understand your work. If you add false statements to a correct argument, you will lose points. Each problem is worth the same amount of points.

(1) Find the length of the curve given by $\mathbf{r}(t) = \langle \frac{1}{2}e^{2t}, 2e^t, 2t \rangle$ from $t = 0$ to $t = 3$.

$$r'(t) = \langle e^{2t}, 2e^t, 2 \rangle$$

$$|r'(t)| = \sqrt{e^{4t} + 4e^{2t} + 4} = \sqrt{(e^{2t}+2)^2} = e^{2t}+2$$

$$\therefore \text{length} = \int_0^3 |r'(t)|dt = \int_0^3 (e^{2t}+2)dt$$

$$= \left[\frac{1}{2}e^{2t} + 2t\right]_0^3 = \frac{1}{2}(e^6 - 1) + 6 = \frac{1}{2}(e^6 + 11)$$

(1)

(2) Calculate the following limits. If they don't exist, explain why.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{(x^2+y^2)^{3/2}}$.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2}$.

(a) Along $y=0$: $\dfrac{x\cdot 0}{(x^2+0^2)^{3/2}} = 0$, limit $= 0$

Along $y=x$: $\dfrac{x\cdot x}{(x^2+x^2)^{3/2}} = \dfrac{x^2}{2^{3/2}\cdot x^3} = \dfrac{1}{2^{3/2}\cdot x}$,

$$\text{limit} = \lim_{x\to 0} \frac{1}{2^{3/2}\cdot x} = \infty$$

$\therefore$ Limit does not exist.

(b) Along $y=0$: $\dfrac{\sin(x\cdot 0)}{x^2} = 0$, limit $= 0$

Along $y=x$: $\dfrac{\sin(x\cdot x)}{x^2} = \dfrac{\sin(x^2)}{x^2}$,

$$\text{limit} = \lim_{x\to 0} \frac{\sin(x^2)}{x^2} = 1.$$

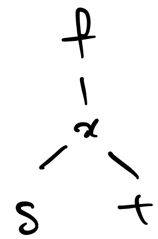
$\therefore$ Limit does not exist.

(2)

(3) (a) Suppose $g = f(s^3 + t^2)$ and $f'(x) = e^x$. Find $\partial g/\partial s$ and $\partial g/\partial t$.

(b) Find $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if $w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \ln s$ and $z = 2r$.

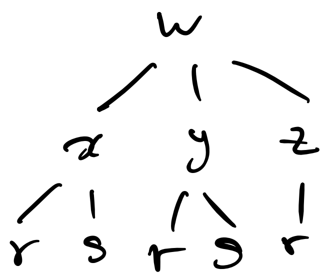
(a)



$$\frac{\partial g}{\partial s} = f'(s^3+t^2) \cdot \frac{\partial}{\partial s}(s^3+t^2) = e^{s^3+t^2} \cdot (3s^2)$$

$$\frac{\partial g}{\partial t} = f'(s^3+t^2) \cdot \frac{\partial}{\partial t}(s^3+t^2) = e^{s^3+t^2} \cdot (2t)$$

(b)



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$= 1 \cdot \frac{1}{s} + 2 \cdot (2r) + 2z \cdot 2 = \frac{1}{s} + 4r + 4(2r) = \frac{1}{s} + 12r$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= 1 \cdot \left(-\frac{r}{s^2}\right) + 2 \cdot \left(\frac{1}{s}\right) = -\frac{r}{s^2} + \frac{2}{s}$$

(3)

(4) Find all absolute maxima and minima of the function $f(x, y) = x^2 + y^2$ on the disk $x^2 - 2x + y^2 - 4y \leq 15$.

$D = \{ (x, y) \mid x^2 - 2x + y^2 - 4y \leq 15 \}$

① Critical point in the domain

$\nabla f = (2x, 2y) = (0, 0) \Rightarrow (x, y) = (0, 0)$.

One can check that $0^2 - 2 \cdot 0 + 0^2 - 4 \cdot 0 < 15$

$\Rightarrow$ it is in D. $f(0,0) = 0$

② On boundary of D

Boundary : $x^2 - 2x + y^2 - 4y = 15$

On this boundary, $f(x, y) = x^2 + y^2 = 2x + 4y + 15$

Let $g(x, y) = x^2 - 2x + y^2 - 4y$

Use Lagrange multiplier :

$\nabla f = \lambda \nabla g$, $(2, 4) = \lambda \cdot (2x - 2, 2y - 4)$

$2 = \lambda(2x - 2) = 2\lambda(x - 1)$, $1 = \lambda(x - 1)$

$4 = \lambda(2y - 4) = 2\lambda(y - 2)$, $2 = \lambda(y - 2)$

$\Rightarrow \lambda = \frac{1}{x - 1}$, $2 = \lambda(y - 2) = \frac{y - 2}{x - 1} \Rightarrow y - 2 = 2(x - 1)$, $y = 2x$.

Plug into constraint : $x^2 - 2x + (2x)^2 - 4(2x) = 5x^2 - 10x = 15$.

$\Rightarrow x^2 - 2x - 3 = 0 = (x + 1)(x - 3)$. $\Rightarrow x = -1$ or 3

$y = -2$ or 6

$f(-1, -2) = 5$, $f(3, 6) = 45$

$\therefore$ abs. max : 45 at $(3, 6)$, abs. min : 0 at $(0, 0)$

(4)

(5) A bed bug is located at the point $(2, -3)$ on a bed whose temperature at (x, y) is $T(x, y) = 20 - 4x^2 - y^2$. Find the equation of the path of the bug as it continuously moves in the direction of maximum temperature increase and sketch the path.

Hint: Use a vector function $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ to represent the position of the bug over time. Find the tangent vector of $\mathbf{r}(t)$ and solve a differential equation to find the path the bug takes.

Direction of maximum temperature increase

$$= \nabla T = (-8x, -2y)$$

Hence we can set $r'(t) = \nabla T(r(t))$

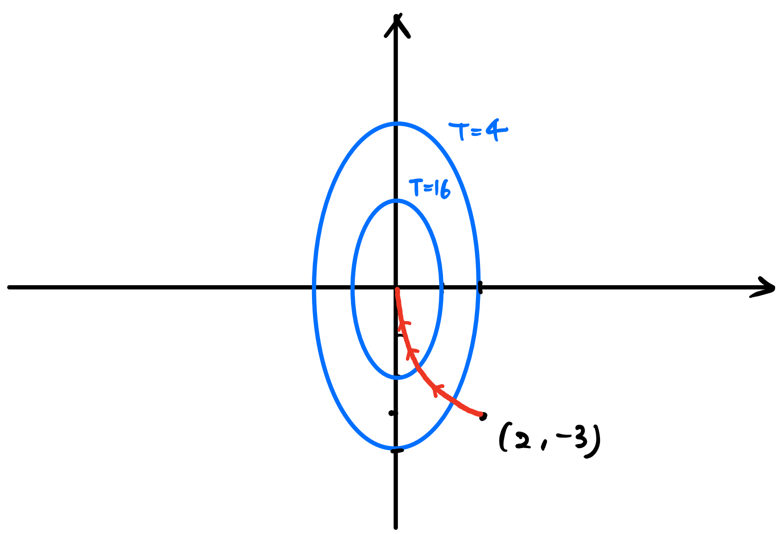
$$\Leftrightarrow (x'(t), y'(t)) = (-8x(t), -2y(t)).$$

$$\Leftrightarrow \begin{cases} x'(t) = -8x(t). & x(0) = 2 \\ y'(t) = -2y(t). & y(0) = -3 \end{cases}$$

Solutions of the above differential equations are

$$\begin{cases} x(t) = 2e^{-8t} \\ y(t) = -3e^{-2t} \end{cases}$$

Since $e^{-8t} = (e^{-2t})^4$, $\frac{x}{2} = \left(\frac{y}{-3}\right)^4 \Leftrightarrow \frac{81}{2}x = y^4$.



(5)

(6) Find and classify all critical points of $f(x, y) = \sin x \sin y$. Use words to describe what the graph of this function looks like.

Use second derivative test.

$f_x = \cos x \sin y$, $f_y = \sin x \cdot \cos y$

$f_{xx} = -\sin x \sin y$, $f_{xy} = \cos x \cos y$, $f_{yy} = -\sin x \sin y$

$D = f_{xx} f_{yy} - f_{xy}^2 = (-\sin x \sin y)(-\sin x \sin y) - (\cos x \cos y)^2$

$= \sin^2 x \sin^2 y - \cos^2 x \cos^2 y$

Critical points :

① $\cos x = 0 \Rightarrow x = (m + \frac{1}{2})\pi$ (m integer), $\sin x \neq 0$

$\Rightarrow \cos y = 0$, $y = (n + \frac{1}{2})\pi$ (n integer)

$\Rightarrow ((m+\frac{1}{2})\pi, (n+\frac{1}{2})\pi)$

$\sin (m+\frac{1}{2})\pi = (-1)^m$, $\cos (m+\frac{1}{2})\pi = 0$

$\Rightarrow f_{xx}((m+\frac{1}{2})\pi, (n+\frac{1}{2})\pi) = -\sin(m+\frac{1}{2})\pi \cdot \sin(n+\frac{1}{2})\pi$

$= -(-1)^m(-1)^n = (-1)^{m+n+1} = \begin{cases} 1 & m+n \text{ odd} \\ -1 & m+n \text{ even} \end{cases}$

$D = \sin^2(m+\frac{1}{2})\pi \sin^2(n+\frac{1}{2})\pi - \cos^2(m+\frac{1}{2})\pi \cos^2(n+\frac{1}{2})\pi$

$= 1 > 0$

$\Rightarrow ((m+\frac{1}{2})\pi, (n+\frac{1}{2})\pi)$, local max when $m+n$ even
(m, n integer) local min when $m+n$ odd

② $\sin y = 0 \Rightarrow y = n\pi$ (n integer)

$\Rightarrow \cos y \neq 0$, $\sin x = 0$

$\Rightarrow x = m\pi$ (m integer) $\Rightarrow (m\pi, n\pi)$

$D = \sin^2(m\pi)\sin^2(n\pi) - \cos^2(m\pi)\cos^2(n\pi) = -1 < 0$

$\Rightarrow (m\pi, n\pi)$ (m, n integer), all saddle points.

It looks like "egg carton".

(6)

(7) Find the equation of the tangent plane of the surface $z = \frac{x^2}{4} + \frac{y^2}{9}$ at the point $(2, 3, 2)$ as well as the equation of the normal line that passes through $(2, 3, 2)$.

$f(x, y) = \frac{x^2}{4} + \frac{y^2}{9}$.

$f_x = \frac{x}{2}$, $f_y = \frac{2y}{9}$, $f_x(2,3) = 1$, $f_y(2,3) = \frac{2}{3}$

$\Rightarrow$ Tangent plane : $z - 2 = 1 \cdot (x-2) + \frac{2}{3}(y-3)$

$\Leftrightarrow z = x + \frac{2}{3}y - 2$

Normal vector $= (f_x, f_y, -1) = (1, \frac{2}{3}, -1)$

$\Rightarrow$ Normal line : $\frac{x-2}{1} = \frac{y-3}{2/3} = \frac{z-2}{-1}$

(7)