Math 53 (Multivariable Calculus), Section 102 & 108
Week 2, Wednesday
Aug 31, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

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1. Find dy/dx and d^2y/dx^2 for the following curves.
   (a) $x = t^3 + 1, y = t^2 - 2$
   (b) $x = e^t + e^{-t}, y = e^{2t} + e^{-2t}$
2. Find the length of the spiral $(e^{-t}\cos t, e^{-t}\sin t)$, where $0 \le t \le 2\pi$. What about $0 \le t < \infty$?
3. Find the area enclosed by the curve $x = t^2, y = t^3 - 3t$.

Solution

1. (a)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2} = \frac{2}{3t}$$

and

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{-\frac{2}{3t^2}}{3t^2} = -\frac{2}{9t^4}.$$

(b)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2e^{2t} - 2e^{-2t}}{e^t - e^{-t}} = 2(e^t + e^{-t})$$

(remark that $a^2 - b^2 = (a-b)(a+b)$, put $a = e^t$ and $b = e^{-t}$), and

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{2(e^t - e^{-t})}{e^t - e^{-t}} = 2.$$

If you noticed that $y = e^{2t} + e^{-2t} = (e^t + e^{-t})^2 - 2 = x^2 - 2$, we can compute more easily: $dy/dx = 2x$ and $d^2y/dx^2 = 2$.

2. We have

$$\frac{dx}{dt} = -e^{-t}\cos t - e^{-t}\sin t, \ \frac{dy}{dt} = -e^{-t}\cos t + e^{-t}\sin t$$

so

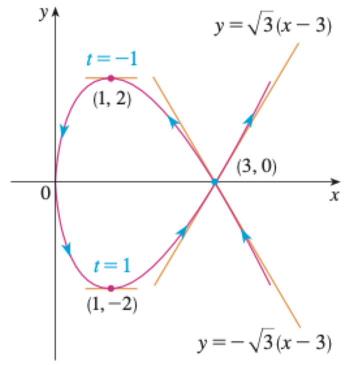
$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= e^{-2t}(\cos^2 t + 2\cos t\sin t + \sin^2 t) + e^{-2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t)\\ &= 2e^{-2t},\\ \text{length} &= \int_0^{2\pi}\sqrt{2e^{-2t}}dt = \sqrt{2}\int_0^{2\pi} e^{-t}dt = \sqrt{2}[-e^{-t}]_0^{2\pi} = \sqrt{2}(1 - e^{-2\pi}).\end{aligned}$$

When $0 \le t < \infty$, we similarly have

$$\text{length} = \int_0^\infty \sqrt{2}e^{-t}dt = \sqrt{2}[-e^{-t}]_0^\infty = \sqrt{2}.$$

In other words, the length is finite even when t varies from 0 to infinity.

3. See the below figure (Figure 1 of 10.2).



The area enclosed by the curve is twice of the upper part area enclosed by the curve and x-axis. The point $(3,0)$ corresponds to $t = -\sqrt{3}$, and the origin $(0,0)$ corresponds to $t = 0$. Hence the total area would be

$$
\begin{aligned}
\text{area} = 2\int_0^3 y\,dx = 2\int_0^{-\sqrt{3}} (t^3 - 3t)(2t)dt \\
&= 4\int_0^{-\sqrt{3}} t^4 - 3t^2 dt \\
&= 4\int_{-\sqrt{3}}^{0} 3t^2 - t^4 dt \\
&= 4\left[t^3 - \frac{1}{5}t^5\right]_{-\sqrt{3}}^{0} \\
&= 4\left(3\sqrt{3} - \frac{9\sqrt{3}}{5}\right) = \frac{24\sqrt{3}}{5}.
\end{aligned}
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