Math 53 (Multivariable Calculus), Section 102 & 108
Week 3, Wednesday
Sep 7, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

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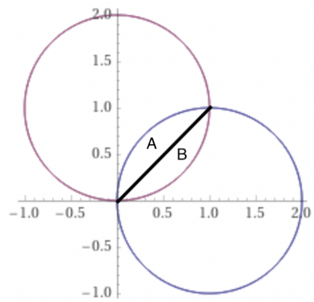
1. Consider two curves $r = 2\cos\theta$ and $r = 2\sin\theta$.
   (a) Find the area of the region that lies inside both curves.
   (b) Find the perimeter of the above region.
2. (a) Sketch a graph of a curve with polar equation $r^2 = \sin 2\theta$.
   (b) Find the area enclosed by the curve.
   (c*) Show that the length of the curve equals to

$$2\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin 2\theta}}$$

Solution

1. (a) Let's find the intersections first. Since both curve passes through origin (i.e. there exists θ such that $2\cos\theta = 0$ or $2\sin\theta = 0$), two curves meets at origin. Also, for $r \neq 0$, $2\cos\theta = 2\sin\theta$, and this gives $\theta = \pi/4 + n\pi$ for $n = \ldots, -1, 0, 1, \ldots$ (you can solve $\tan\theta = 1$). Every n corresponds to a single point $(r, \theta) = (\sqrt{2}, \pi/4) = (-\sqrt{2}, 5\pi/4)$. Hence there are two intersections: $(0, 0)$ and $(\sqrt{2}, \pi/4)$. Now let's see



the above graph of two curves. The area that lies inside both curves equals to the area of A and B. A is a region bounded by the curve $r = 2\cos\theta$ and the ray $\theta = \pi/4$ and $\theta = \pi/2$ (y-axis), hence

$$\text{area}(A) = \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos\theta)^2 d\theta = \int_{\pi/4}^{\pi/2} 2\cos^2\theta d\theta = \int_{\pi/4}^{\pi/2} (1+\cos 2\theta) d\theta = \left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/2} = \frac{\pi}{4} - \frac{1}{2}$$

Similarly, we can check that $\text{area}(A) = \int_0^{\pi/4} \frac{1}{2}(2\sin\theta)^2 d\theta = \frac{\pi}{4} - \frac{1}{2}$, so the total area is $\frac{\pi}{2} - 1$.

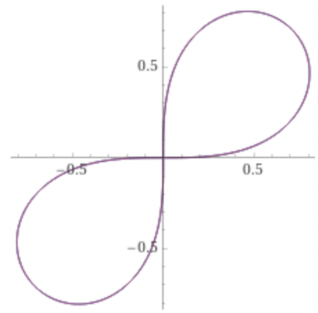
(b) The perimeter of the above region is the sum of the piece of the curve $r = 2\cos\theta$ that bounds A and another piece of the curve $r = 2\sin\theta$ that bounds B. The length of the first piece is

$$\text{length}(A) = \int_{\pi/4}^{\pi/2} \sqrt{(2\cos\theta)^2 + (2\sin\theta)^2} d\theta = \int_{\pi/4}^{\pi/2} 2 d\theta = \frac{\pi}{2}$$

and the other piece also has length $\text{length}(B) = \int_0^{\pi/4} 2d\theta = \pi/2$. Hence the perimeter is π.

Remark. Once you now that two curves are circles, you may be able to anwer the questions without any integrals at all.

2. (a) Here's a graph.



(b)

$$
\begin{aligned}
\text{area} &= 2\int_0^{\pi/2} \frac{1}{2} r^2 d\theta \\
&= \int_0^{\pi/2} \sin 2\theta d\theta \\
&= \left[-\frac{1}{2}\cos 2\theta\right]_0^{\pi/2} = 1
\end{aligned}
$$

(c) By deriving both side of $r^2 = \sin 2\theta$ with respect to θ, we get

$$
2r\frac{dr}{d\theta} = 2\cos 2\theta \Rightarrow \frac{dr}{d\theta} = \frac{\cos 2\theta}{r}
$$

Hence the length is

$$
\begin{aligned}
\text{length} &= 2\int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \\
&= 2\int_0^{\pi/2} \sqrt{\sin 2\theta + \frac{\cos^2 2\theta}{\sin 2\theta}} d\theta \\
&= 2\int_0^{\pi/2} \sqrt{\frac{\sin^2 2\theta + \cos^2 2\theta}{\sin 2\theta}} d\theta \\
&= 2\int_0^{\pi/2} \frac{1}{\sqrt{\sin 2\theta}} d\theta.
\end{aligned}
$$

Remark. The integral cannot be expressed in a closed form.