Math 53 (Multivariable Calculus), Section 102 & 108
Week 4, Monday
Sep 12, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

1. Identify the type of conic section whose equation is given and find vertices and foci.
   (a) $\frac{x^2}{4} + y^2 - x - 2y + 1 = 0$
   (b) $y^2 - x - 2y + 1 = 0$
2. Let $\mathbf{a} = \langle 1, 1 \rangle$ and $\mathbf{b} = \langle 1, 0 \rangle$.
   (a) Let $\mathbf{c} = \langle 5, 2 \rangle$. Show, by means of a sketch, that there are scalars s and t such that $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$.
   (b) Find the values of s and t.
   (c) Can you find s, t when $\mathbf{a}, \mathbf{c}$ remains the same but $\mathbf{b} = \langle -2, -2 \rangle$?

Solution

1. (a) By completing the square, we get

$$\frac{x^2}{4} - x + y^2 - 2y + 1 = \frac{1}{4}(x-2)^2 - 1 + (y-1)^2 = 0 \Leftrightarrow \frac{(x-2)^2}{2^2} + (y-1)^2 = 1$$

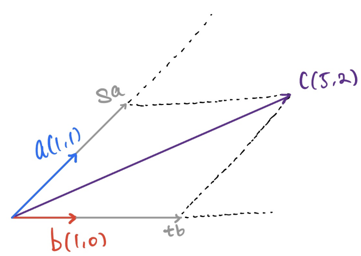
Hence it is an ellipse where the ellipse $x^2/2^2 + y^2 = 1$ is shifted to the right by 2 and to the upward by 1. Since the foci of the original ellipse are $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$ $(a = 2, b = 1, c = \sqrt{a^2 - b^2} = \sqrt{3})$, the shifted ellipse's foci are $(2 + \sqrt{3}, 1)$ and $(2 - \sqrt{3}, 1)$. The original ellipse's vertices are $(2, 0)$ and $(-2, 0)$, so the shifted ellipse's vertices are $(4, 1)$ and $(0, 1)$.

(b) By completing the square, we get

$$-x + y^2 - 2y + 1 = -x + (y-1)^2 = 0 \Leftrightarrow (y-1)^2 = x = 4 \cdot \frac{1}{4} \cdot x.$$

Hence it is a parabola where the parabola $y^2 = x$ is shifted to the upward by 1. We have $p = 1/4$ and the original parabola's focal point is $(1/4, 0)$ and the vertex is $(0, 0)$. Then our shifted parabola's focal point is $(1/4, 1)$ and the vertex is $(0, 1)$.

2. (a) As you can see below, by scaling $\mathbf{a}$ and $\mathbf{b}$ suitably, they may adds up to $\mathbf{c}$.



(b) The equation becomes

$$\langle 5, 2\rangle = s\langle 1, 1\rangle + t\langle 1, 0\rangle \Leftrightarrow \begin{cases} 5 = s + t \\ 2 = s \end{cases}$$

and we get $s = 2, t = 5 - s = 3$.

(c) We can observe that $\langle 1, 1\rangle$ and $\langle -2, 2\rangle$ has the same direction, but $\mathbf{c} = \langle 5, 2\rangle$ is not pointing that direction. So intuitively, multiplying s, t to each $\mathbf{a}$ and (new) $\mathbf{b}$ and adding them still pointing the same direction as $\mathbf{a}$ and $\mathbf{b}$, so they cannot produce $\mathbf{c}$. In fact, we have

$$s\langle 1, 1\rangle + t\langle -2, 2\rangle = \langle s - 2t, s - 2t\rangle$$

so that such vectors have identical components, while $\mathbf{c} = \langle 5, 2\rangle$ doesn't. Hence we can't (there's no such s and t).