Math 53 (Multivariable Calculus), Section 102 & 108
Week 4, Wednesday
Sep 14, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

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1. Find unit vectors that are orthogonal to $\langle 1, 1, 0\rangle$ and $\langle 1, 0, 1\rangle$.
2. Let $\mathbf{r} = \langle x, y\rangle$, $\mathbf{a} = \langle 1, 1\rangle$, $\mathbf{b} = \langle 1, -1\rangle$. Show that the vector equation $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$ represents a circle.
3. For three points $P(1, 0, 1), Q(-2, 1, 3), R(4, 2, 5)$, find a nonzero vector orthogonal to the plane through the points P, Q, and R. Find the area of triangle PQR.

Solution

1. There are several ways to do this: using dot product or cross product.

   (a) (Using dot product) Let $\mathbf{u} = \langle x, y, z\rangle$ be a such vector. Since it is orthogonal to $\mathbf{a} = \langle 1, 1, 0\rangle$ and $\mathbf{b} = \langle 1, 0, 1\rangle$, we have

   $$\mathbf{u}\cdot\mathbf{a} = 0 \Leftrightarrow \langle x, y, z\rangle \cdot \langle 1, 1, 0\rangle = x + y = 0$$
   $$\mathbf{u}\cdot\mathbf{b} = 0 \Leftrightarrow \langle x, y, z\rangle \cdot \langle 1, 0, 1\rangle = x + z = 0$$

   Using these equations, we get $y = -x$ and $z = -x$, so that we can express $\mathbf{u}$ only in terms of x: $\mathbf{u} = \langle x, -x, -x\rangle$. The length of $\mathbf{u}$ is

   $$|\mathbf{u}| = \sqrt{x^2 + (-x)^2 + (-x)^2} = \sqrt{3}|x| = 1,$$

   and this results $x = \pm 1/\sqrt{3}$, $\mathbf{u} = \langle 1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3}\rangle, \langle -1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\rangle$.

   (b) (Using cross product) Since the vector $\mathbf{u}$ is orthogonal to both vectors, it would have the same direction as $\mathbf{a}\times\mathbf{b}$, which is $\langle 1, -1, -1\rangle$. Then we can find unit vectors that have same direction with given vector by dividing a vector with its length: $\sqrt{3}$. Note that the opposite of the vector $\frac{1}{\sqrt{3}}\langle 1, -1, -1\rangle$ also a unit vector orthogonal to given vectors.

2. $$\begin{aligned}(\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{b}) = \langle x-1, y-1\rangle \cdot \langle x-1, y+1\rangle &= (x-1)^2 + y^2 - 1 = 0 \\ &\Leftrightarrow (x-1)^2 + y^2 = 1\end{aligned}$$

   Hence it represents a circle of radius 1 centered at $(1, 0)$.

3. We can find such a vector by taking a cross-product of any two vectors lives in the plane. For example, we can choose $\overrightarrow{PQ}$ and $\overrightarrow{PR}$:

   $$\begin{aligned}\overrightarrow{PQ} &= \langle (-2)-1, 1-0, 3-1\rangle = \langle -3, 1, 2\rangle \\ \overrightarrow{PR} &= \langle 4-1, 2-0, 5-1\rangle = \langle 3, 2, 4\rangle \\ \overrightarrow{PQ}\times\overrightarrow{PR} &= \langle 1\cdot 4 - 2\cdot 2, 2\cdot 3 - (-3)\cdot 4, (-3)\cdot 2 - 1\cdot 3\rangle = \langle 0, 18, -9\rangle.\end{aligned}$$

   The area of the triangle PQR is the half of the area of parallelogram formed by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Hence the area of the triangle is

   $$\frac{1}{2}|\overrightarrow{PQ}\times\overrightarrow{PR}| = \frac{1}{2}\sqrt{0^2 + 18^2 + (-9)^2} = \frac{\sqrt{405}}{2} = \frac{9\sqrt{5}}{2}.$$