**Math 53 (Multivariable Calculus), Section 102 & 108**
**Week 5, Friday**
**Sep 23, 2022**
**For the other materials: seewoo5.github.io/teaching/2022Fall**

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1. Show that the curve with parametric equations $x = t\cos t, y = t\sin t, z = t^2$ lies on the elliptic paraboloid $x^2 + y^2 = z$, and use this fact to sketch the curve.

2. Show that the curve with parametric equations $x(t) = t^2 - 1, y(t) = -t + 1, z(t) = -t^2 + t + 1$ lies on a plane. Find an equation of the plane.

3. Find a vector function that represents the curve of intersection of the hyperboloid $z = x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$.

**Solution**

1. For a point on a curve, we have

$$x^2 + y^2 = (t\cos t)^2 + (t\sin t)^2 = t^2(\cos^2 t + \sin^2 t) = t^2 = z$$

   so it lies on the elliptic paraboloid. Graph can be found on the next page.

2. Let's assume that the equation of the plane is given by $ax + by + cz + d = 0$ for some $a, b, c, d$. Then, for all $t$, we should have

$$a(t^2 - 1) + b(-t + 1) + c(-t^2 + t + 1) + d = (a - c)t^2 + (-b + c)t + (-a + b + c + d) = 0$$

   From this, we have $a - c = 0 \Leftrightarrow a = c$, and $-b + c = 0 \Leftrightarrow b = c$. Then we get $-a + b + c + d = 0 = -a + a + a + d = a + d \Leftrightarrow d = -a$. It means that our equation of the plane is

$$ax + by + cz + d = ax + ay + az - a = a(x + y + z - 1) = 0,$$

   and any choice of $a$ gives the (essentially) same equation, for example, $x + y + z - 1 = 0$.

   Another approach is using (any) 3 points on a curve and find a plane that contains all three points. For example, we can choose points corresponding to $t = -1, 0, 1$ and we get $A(0, -2, -1), B(-1, 1, 1), C(0, 0, 1)$. Then we can find a normal vector using cross-product:

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \langle -1, 3, 2\rangle \times \langle 0, 2, 2\rangle = \langle 2, 2, 2\rangle$$

   so an equation of the plane would have a form of $2x + 2y + 2z + d = 0$ for some $d$. Now plugging any point (for example, $C$) into the equation gives $d = -2$ and we get $2x + 2y + 2z - 2 = 0 \Leftrightarrow x + y + z - 1 = 0$. It is important to check that this plane is *the* plane that we are looking for: in other words, all the other points on the curve with different $t$'s should be also on the plane. This follows from the direct calculation as

$$(t^2 - 1) + (-t + 1) + (-t^2 + t + 1) - 1 = 0.$$

3. First, we can express $x$ and $y$ in terms of a single parameter $t$ as $x = \cos t, y = \sin t$. Then we have $z = x^2 - y^2 = \cos^2 t - \sin^2 t = \cos 2t$, and this gives a vector function

$$\mathbf{r}(t) = (\cos t, \sin t, \cos 2t)$$

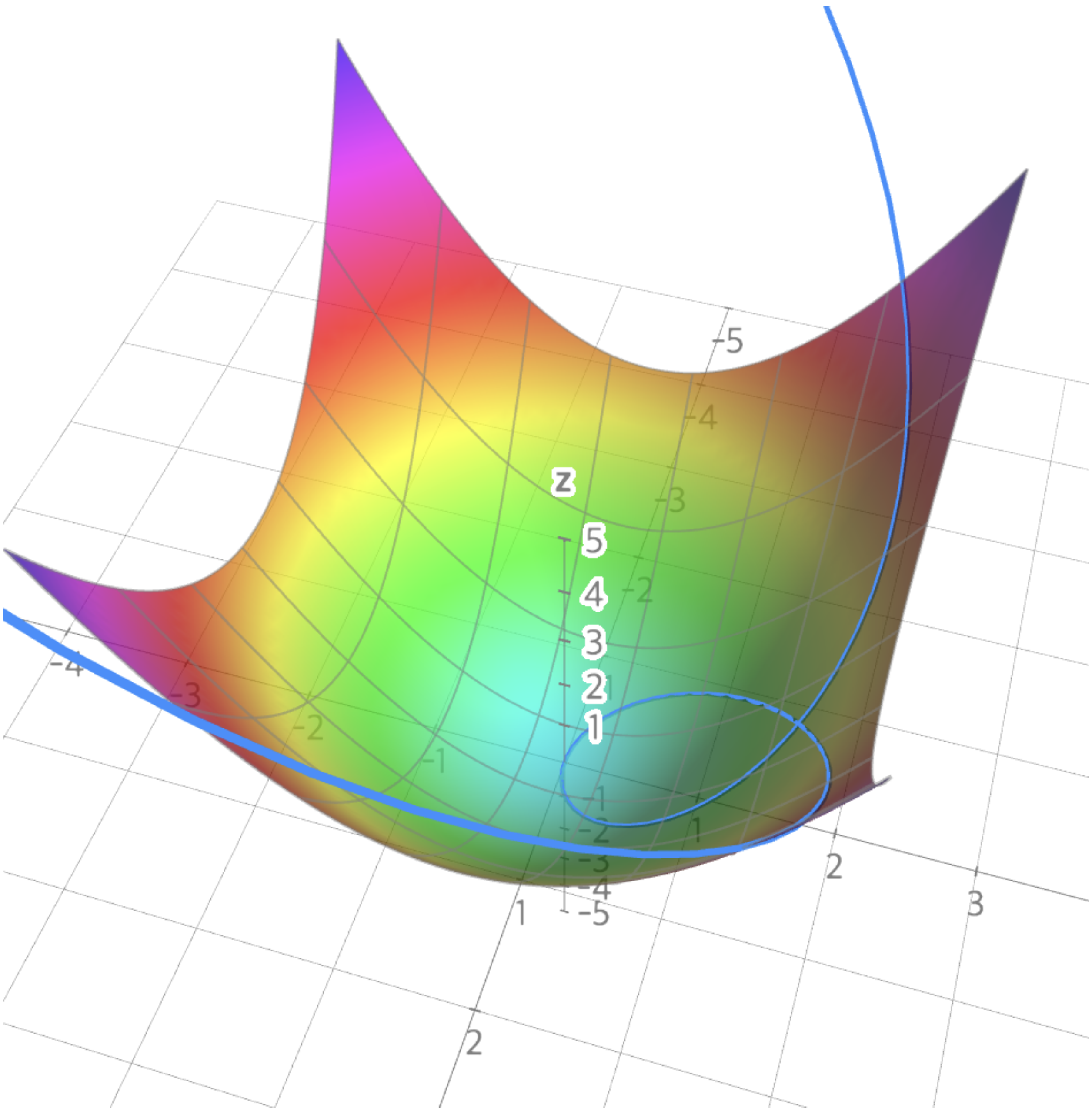


Figure 1: Graph of $z = x^2 + y^2$ and the curve on it.