Math 53 (Multivariable Calculus), Section 102 & 108
Week 6, Monday
Sep 26, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

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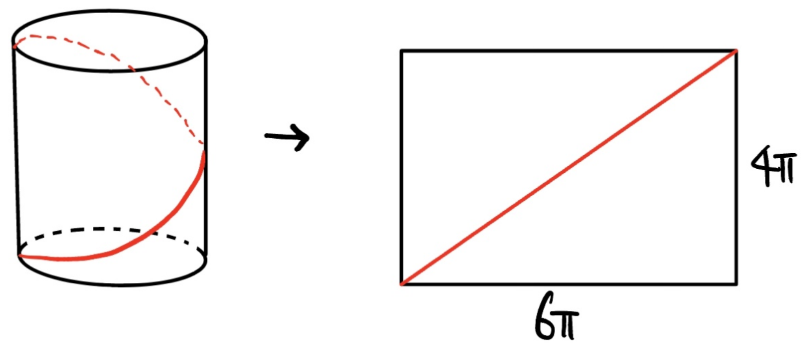
1. Find length of the curve $\mathbf{r}(t) = 2t\mathbf{i} + 3\cos t\mathbf{j} + 3\sin t\mathbf{k}$ for $0 \leq t \leq 2\pi$.
2. The position of a particle is given by $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t\rangle$. When is the speed minimum?
3. Assume that a particle moves with a constant speed. Show that its velocity and acceleration are always orthogonal. (Hint: consider $\frac{d}{dt}|\mathbf{r}'(t)|^2$.)

Solution

1. We have

$$\text{length} = \int_0^{2\pi} |\mathbf{r}'(t)| dt = \int_0^{2\pi} \sqrt{2^2 + (-3\sin t)^2 + (3\cos t)^2} dt = \int_0^{2\pi} \sqrt{13} dt = 2\pi\sqrt{13}.$$

We can also find the length in the following way. The curve, which is a helix, is on the cylinder $y^2 + z^2 = 9$. If you cut the cylinder vertically and spread it, the helix become a straight line, which is a diagonal of a rectangle with side lengths 6π and 4π. Hence the answer is $\sqrt{(6\pi)^2 + (4\pi)^2} = 2\pi\sqrt{13}$.



2. The square of the speed is

$$|\mathbf{r}'(t)|^2 = |\langle 2t, 5, 2t - 16\rangle|^2 = 4t^2 + 25 + (2t - 16)^2 = 8t^2 - 64t + 281.$$

Its derivative is $\frac{d}{dt}|\mathbf{r}'(t)|^2 = \frac{d}{dt}(8t^2 - 64t + 281) = 16t - 64$, so it has a minimum value when $t = 4$.

3. We have

$$\frac{d}{dt}|\mathbf{r}'(t)|^2 = \frac{d}{dt}(\mathbf{r}'(t) \cdot \mathbf{r}'(t)) = \mathbf{r}''(t) \cdot \mathbf{r}'(t) + \mathbf{r}'(t) \cdot \mathbf{r}''(t) = 2\mathbf{r}'(t) \cdot \mathbf{r}''(t).$$

Hence if the speed $|\mathbf{r}'(t)|$ is constant, then its derivative is zero and the above equation gives $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 0$. So the velocity and acceleration are orthogonal.