Math 53 (Multivariable Calculus), Section 102 & 108
Week 6, Wednesday
Sep 28, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

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1. Graph the functions.
   (a) $f(x, y) = \sqrt{x^2 + y^2}$
   (b) $f(x, y) = e^{\sqrt{x^2+y^2}}$
   (c) $f(x, y) = \ln(x^2 + y^2)$

   What do they have in common? Can you also graph the following function?

$$f(x, y) = \sqrt{1 - (\sqrt{x^2 + y^2} - 2)^2}$$

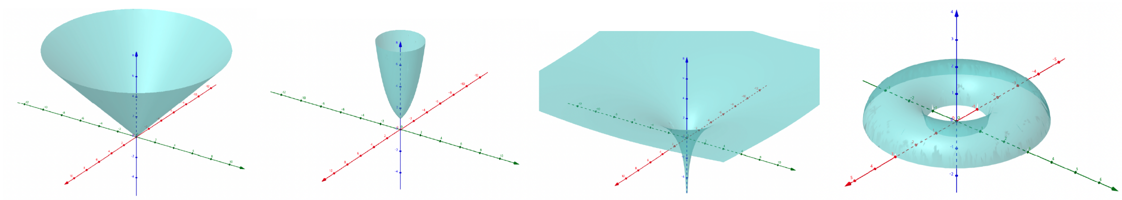
2. Consider the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y^3}{x^4 + y^6}$$

   (a) Show that the limit along line $y = mx$ exists and the same for all m.
   (b) Show that, for any positive integer n, the limit along $y = mx^n$ exists and the same for all m.
   (c) Show that the limit does not exist.

Solution

1. All of them has a form of $f(x, y) = g(\sqrt{x^2 + y^2})$ for some g. Since $r = \sqrt{x^2 + y^2}$ is a distance from (x, y) to $(0, 0)$, such functions only depends on the distance to the origin. In other words, all the points on a circle centered at origin have same values. Graph of such functions can be obtained by rotating a graph of $z = g(r)$ (for $r \geq 0$) around z-axis. The last function corresponds to $g(r) = \sqrt{1 - (r - 2)^2} \Rightarrow (r - 2)^2 + g(r)^2 = 1$. Hence it is a rotation of upper semi-circle (not the whole circle) of radius 1 and centered at $(2, 0)$, and it becomes a half of donut-like surface.



2. (a) Along $y = mx$, we have

$$\lim_{x \to 0} \frac{x^2 (mx)^3}{x^4 + (mx)^6} = \lim_{x \to 0} \frac{m^3 x^5}{x^4 + m^6 x^6} = \lim_{x \to 0} \frac{m^3 x}{1 + m^6 x^2} = 0.$$

(b) Along $y = mx^n$, we have

$$\lim_{x \to 0} \frac{x^2 (mx^n)^3}{x^4 + (mx^n)^6} = \lim_{x \to 0} \frac{m^3 x^{3n+2}}{x^4 + m^6 x^{6n}} = \lim_{x \to 0} \frac{m^3 x^{3n-2}}{1 + m^6 x^{6n-4}} = 0.$$

(c) Along $y = x^{2/3}$, we have

$$\lim_{x \to 0} \frac{x^2 (x^{2/3})^3}{x^4 + (x^{2/3})^6} = \lim_{x \to 0} \frac{x^4}{x^4 + x^4} = \frac{1}{2},$$

which is different from the above limits (which are 0). Hence the limit does not exist.