Math 53 (Multivariable Calculus), Section 102 & 108
Week 7, Monday
Oct 3, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

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1. Show that

$$c(x,t) = \frac{1}{\sqrt{4\pi D t}} e^{-x^2/(4Dt)}$$

is a solution of the *diffusion equation*

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}.$$

2. Let

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

   (a) Compute $f_x(0,0), f_y(0,0), f_{xy}(0,0), f_{yx}(0,0)$. Check that $f_{xy}(0,0) = f_{yx}(0,0)$.

   (b) Show that both f_{xy} and f_{yx} are not continuous at $(0,0)$.

Solution

1.

$$\begin{aligned}\frac{\partial c}{\partial t} &= \frac{1}{\sqrt{4\pi D}} \cdot -\frac{1}{2} t^{-3/2} e^{-x^2/(4Dt)} + \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/(4Dt)} \cdot \frac{x^2}{4Dt^2} \\ &= \left(-\frac{1}{2\sqrt{4\pi Dt^3}} + \frac{x^2}{4\sqrt{4\pi D^3 t^5}} \right) e^{-x^2/(4Dt)}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial c}{\partial x} &= \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/(4Dt)} \cdot -\frac{x}{2Dt}, \\ \frac{\partial^2 c}{\partial x^2} &= \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/(4Dt)} \cdot \left(-\frac{x}{2Dt} \right)^2 + \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/(4Dt)} \cdot -\frac{1}{2Dt} \\ &= \left(-\frac{1}{2\sqrt{4\pi D^3 t^3}} + \frac{x^2}{4\sqrt{4\pi D^5 t^5}} \right) e^{-x^2/(4Dt)}\end{aligned}$$

so $\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}$.

2. (a) Since $f(x,0) = 0$, we have $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$. Similarly, we get $f_y(0,0) = 0$. For $(x,y) \neq (0,0)$, we have

$$\begin{aligned}f_x(x,y) &= \frac{2xy^2(x^2+y^2) - x^2y^2(2x)}{(x^2+y^2)^2} = \frac{2xy^4}{(x^2+y^2)^2} \\ f_y(x,y) &= \frac{2x^2y(x^2+y^2) - x^2y^2(2y)}{(x^2+y^2)^2} = \frac{2x^4y}{(x^2+y^2)^2}\end{aligned}$$

and $f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,h)-f_x(0,0)}{h} = 0$, and similarly we have $f_{yx}(0,0) = 0$. Hence both are the same.

(b) For $(x,y) \neq (0,0)$, we have

$$\begin{aligned}f_{xy}(x,y) &= \frac{8xy^3(x^2+y^2)^2 - 2xy^4 \cdot 2(2y)(x^2+y^2)}{(x^2+y^2)^4} = \frac{8x^5y^3 + 8x^3y^5}{(x^2+y^2)^4} = \frac{8x^3y^3}{(x^2+y^2)^3} \\ f_{yx}(x,y) &= \frac{8x^3y(x^2+y^2)^2 - 2x^4y \cdot 2(2x)(x^2+y^2)}{(x^2+y^2)^4} = \frac{8x^3y^5 + 8x^5y^3}{(x^2+y^2)^4} = \frac{8x^3y^3}{(x^2+y^2)^3}\end{aligned}$$

and two functions are the same when $(x,y) \neq (0,0)$. If we consider the limit $(x,y) \to (0,0)$ along $y = mx$, we have

$$\lim_{x\to 0} f_{xy}(x,mx) = \lim_{x\to 0} \frac{8m^3x^6}{(1+m^2)^3x^6} = \frac{8m^6}{(1+m^2)^3}$$

which gives different values for different m's. Hence the limit $\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ does not exist.