Math 53 (Multivariable Calculus), Section 102 & 108
Week 7, Wednesday
Oct 5, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

---

1. Let $f(x, y) = 1 + x\ln(xy - 5)$.
   (a) Find the linearization of $f(x, y)$ at $(2, 3)$.
   (b) Using (a), estimate $f(2.01, 2.99)$.
2. Let $z = x^2 - y^2$, $x = e^t + e^{-t}$, $y = e^t - e^{-t}$. Find dz/dt.
3. Let $z = x^2 + y^2 + \frac{x}{y}$, $x = r\cos\theta$, $y = r\sin\theta$. Find $\partial z/\partial r$ and $\partial z/\partial\theta$.

Solution

1. (a) We have

$$\begin{aligned}
f(2,3) &= 1 \\
f_x(x,y) &= \ln(xy-5) + \frac{xy}{xy-5} \Rightarrow f_x(2,3) = 6 \\
f_y(x,y) &= \frac{x^2}{xy-5} \Rightarrow f_y(2,3) = 4
\end{aligned}$$

so the linearization of $f(x,y)$ at $(2,3)$ becomes $L(x,y) = 1 + 6(x-2) + 4(y-3)$

(b) $f(2.01, 2.99) \approx L(2.01, 2.99) = 1 + 6(0.01) + 4(-0.01) = 1.02$. Note that the true value is $f(2.01, 2.99) = 1.0198...$

2. By chain rule, we have

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = 2x(e^t - e^{-t}) - 2y(e^t + e^{-t}) \\
&= 2(e^t + e^{-t})(e^t - e^{-t}) - 2(e^t - e^{-t})(e^t + e^{-t}) = 0
\end{aligned}$$

In fact, we have $z = x^2 - y^2 = (e^t + e^{-t})^2 - (e^t - e^{-t})^2 = 4$, which is a constant function and $dz/dt = 0$.

3. By chain rule, we have

$$\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \left(2x + \frac{1}{y}\right)\cos\theta + \left(2y - \frac{x}{y^2}\right)\sin\theta \\
&= \left(2r\cos\theta + \frac{1}{r\sin\theta}\right)\cos\theta + \left(2r\sin\theta - \frac{\cos\theta}{r\sin^2\theta}\right)\sin\theta = 2r, \\
\frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \left(2x + \frac{1}{y}\right)(-r\sin\theta) + \left(2y - \frac{x}{y^2}\right)(r\cos\theta) \\
&= \left(2r\cos\theta + \frac{1}{r\sin\theta}\right)(-r\sin\theta) + \left(2r\sin\theta - \frac{\cos\theta}{r\sin^2\theta}\right)(r\cos\theta) \\
&= -1 - \cot^2\theta = -\csc^2\theta.
\end{aligned}$$

Also, we can directly plug in $x = r\cos\theta$ and $y = r\sin\theta$ so that $z = r^2 + \cot\theta$ and compute $\partial z/\partial r$ and $\partial z/\partial\theta$, which gives the same answer.