Math 53 (Multivariable Calculus), Section 102 & 108
Week 7, Friday
Oct 7, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

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1. Find the directions in which $f(x, y) = 4y\sqrt{x}$
   (a) increases most rapidly at $(4, 1)$
   (b) decreases most rapidly at $(4, 1)$
   (c) has zero change at $(4, 1)$
2. Find all points at which the direction of fastest change of the function $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\mathbf{i} + \mathbf{j}$.
3. Find equations of the tangent plane and the normal line to the surface $xy^2z^3 = 12$ at a point $(3, 2, 1)$.

Solution

1. First, gradient of the function at $(4,1)$ is

$$\nabla f = \left\langle \frac{2y}{\sqrt{x}}, 4\sqrt{x} \right\rangle \Rightarrow \nabla f(4,1) = \langle 1, 8 \rangle.$$

   (a) the direction where the function increases most rapidly is the same as the direction of the gradient, which is $\langle 1, 8 \rangle$,

   (b) the direction where the function decreases most rapidly is the same as the opposite direction of the gradient, which is $\langle -1, -8 \rangle$,

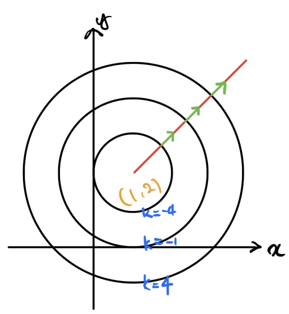
   (c) the direction where the function has zero change is the same as the direction orthogonal to the gradient, which is $\langle 8, -1 \rangle$ and $\langle -8, 1 \rangle$,

2. The direction of fastest change equals to the direction of gradient. The gradient of f is $\nabla f = \langle f_x, f_y \rangle = \langle 2x-2, 2y-4 \rangle$, and we want this vector to be the same direction as $\mathbf{i}+\mathbf{j}$. We can write as $\nabla f = c(\mathbf{i}+\mathbf{j}) = \langle c, c \rangle$ for some $c > 0$. (Note that c can't be negative since we want gradient to be the same direction as $\mathbf{i}+\mathbf{j}$) It becomes

$$\begin{cases} 2x - 2 = c \\ 2y - 4 = c \end{cases} \quad \Rightarrow 2x - 2 = 2y - 4 \Leftrightarrow y = x + 1,$$

   and from $\nabla f = \langle 2x-2, 2y-4 \rangle = \langle c, c \rangle$, $c > 0$ is equivalent to $2x - 2 > 0 \Leftrightarrow x > 1$. Hence the set of the points becomes a *half* line represented by $y = x + 1$ and $x > 1$.

   One can also do this by considering level sets. The level sets of the function are circles centered at $(1,2)$, and the gradients at each point are the vectors orthogonal to the level sets (the circles). Using this, you can figure out that the set of points where the gradient is parallel to $\mathbf{i}+\mathbf{j}$ is $y = x+1$ and $x > 1$.



3. Normal vector can be found using gradient: the gradient of $F(x,y,z) = xy^2z^3$ equals to $\nabla F = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$, and the normal vector at the point is $\vec{n} = \nabla F(3,2,1) = \langle 4, 12, 36 \rangle$. Hence the equations for the tangent plane and the normal line is

$$4(x-3) + 12(y-2) + 36(z-1) = 0 \Leftrightarrow x + 3y + 9z = 18$$

$$\frac{x-3}{4} = \frac{y-2}{12} = \frac{z-1}{36} \Leftrightarrow x - 3 = \frac{y-2}{3} = \frac{z-1}{9}$$