Math 53 (Multivariable Calculus), Section 102 & 108
Week 8, Monday
Oct 10, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

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1. Find the local maximum and minimum values and saddle point(s) of the function $f(x, y) = x^2 + y^4 + 2xy$.
2. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.
3. Find the side lengths of the rectangular box with volume 8m^3 that has minimal surface area.

Solution

1. We have $f_x = 2x + 2y$ and $f_y = 4y^3 + 2x$, so the critical points (x, y) satisfy $2x + 2y = 0 \Leftrightarrow y = -x$ and $4y^3 + 2x = 0 = -4x^3 + 2x = 2x(1 - 2x^2) = 2x(1 - \sqrt{2}x)(1 + \sqrt{2}x)$, so the critical points are $(0, 0)$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Also, second derivatives of f are

$$f_{xx} = 2, \quad f_{xy} = 2, \quad f_{yy} = 8y^2.$$

   At these points, we have

$$\begin{aligned} D(0,0) &= -4 \Rightarrow \text{Saddle point} \\ D\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) &= 4, f_{xx} > 0 \Rightarrow \text{Local minimum} \\ D\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) &= 4, f_{xx} > 0 \Rightarrow \text{Local minimum.} \end{aligned}$$

2. For a point (x, y, z) on a cone, the distance between it an $(4, 2, 0)$ is

$$\begin{aligned} \sqrt{(x-4)^2 + (y-2)^2 + z^2} &= \sqrt{(x-4)^2 + (y-2)^2 + x^2 + y^2} \\ &= \sqrt{2x^2 + 2y^2 - 8x - 4y + 20}. \end{aligned}$$

   Let $f(x, y) = 2x^2 + 2y^2 - 8x - 4y + 20$ be the square of the distance (to make the computation simpler), and we are going to find x, y that minimizes f. Since $f_x = 4x - 8$ and $f_y = 4y - 4$, the only critical point of f is $(2, 1)$, and by the second derivative test, $D(2, 1) = f_{xx}(2, 1)f_{yy}(2, 1) - f_{xy}(2, 1)^2 = 16 > 0$ and $f_{xx}(2, 1) = 4 > 0$ shows that $f(x, y)$ has a minimum at $(2, 1)$, and the corresponding points on the cone are $(2, 1, \sqrt{5})$ and $(2, 1, -\sqrt{5})$. You can also deduce the same conclusion by completing the square: $f(x, y) = 2(x - 2)^2 + 2(y - 1)^2 + 10$.

3. Let x, y, z be the side lengths of the box, so that $xyz = 8$. Let $f(x, y)$ be the surface area of the box in terms of x, y, which is

$$2xy + 2yz + 2zx = 2xy + \frac{16}{x} + \frac{16}{y}.$$

   From $f_x = 2y - \frac{16}{x^2}$ and $f_y = 2x - \frac{16}{y^2}$, we have $x^2y = 8$ and $xy^2 = 8$, so $x^2 = xy^2 \Leftrightarrow xy(x - y) = 0$. Hence $x = y$ (since $x, y > 0$) and $x^3 = 8 \Rightarrow x = 2$ and $y = 2$. From $xyz = 8$, we also have $z = 2$ and the side lengths of the box is $2, 2, 2$.