Math 53 (Multivariable Calculus), Section 102 & 108
Week 8, Wednesday
Oct 12, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

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1. Find the absolute maximum and minimum of the function $f(x, y) = x^2 + y^2 - 2x + 1$ on the disk $D = \{(x, y)|x^2 + y^2 \leq 4\}$. Can you find them without using partial derivatives?

2. Let $H = -x \ln x - y \ln y - z \ln z$.

   (a) Assume that x, y, z are positive and $x + y + z = 1$. Express H as a function in x and y only.

   (b) Find maximum value of H. For what values of x, y, z does it occur?

   (c) Use Lagrange multiplier to derive the same conclusion without expressing H as a two variable function.

   The function H is called *Shannon's entropy*.

Solution

1. First, the partial derivatives of f are $f_x = 2x - 2$ and $f_y = 2y$, so the unique critical point is $(1, 0)$ which is in D. Since $f_{xx} = 2 > 0$, $f_{yy} = 2$ and $f_{xy} = 0$, we have $D = f_{xx}f_{yy} - f_{xy}^2 > 0$ and so f has a local minimum 0 at $(1, 0)$. On the boundary of D, we have $x^2 + y^2 = 4$ (the circle of radius 2) and $f(x, y) = x^2 + y^2 - 2x + 1 = -2x + 5$, so it has a minimum 1 at $(2, 0)$ (where x is maximized) and maximum 9 at $(-2, 0)$ (where x is minimized) on boundary. Hence the absolute maximum is 9 and minimum is 0.

   If you noticed that $f(x, y) = (x - 1)^2 + y^2$, then it is a square of the distance between (x, y) and $(1, 0)$. Using this, you can graphically find that the maximum and minimum of f are attained at $(-2, 0)$ and $(1, 0)$, respectively.

2. (a) We have $z = 1 - x - y$, so $H = -x \ln x - y \ln y - (1 - x - y) \ln(1 - x - y)$.

   (b) The partial derivatives of H are

$$H_x = -\ln x + \ln(1 - x - y)$$
$$H_y = -\ln y + \ln(1 - x - y)$$

   and the critical point of H are (x, y) where

$$-\ln x + \ln(1 - x - y) = 0 \Leftrightarrow x = 1 - x - y \Leftrightarrow 2x + y = 1$$
$$-\ln y + \ln(1 - x - y) = 0 \Leftrightarrow y = 1 - x - y \Leftrightarrow x + 2y = 1$$

   Once we solve the equation (you can get $y = 1 - 2x$ from the first equation and plug it into the second equation), we have $x = y = 1/3$. The second derivatives of H are

$$H_{xx} = -\frac{1}{x} - \frac{1}{1 - x - y}$$
$$H_{yy} = -\frac{1}{y} - \frac{1}{1 - x - y}$$
$$H_{xy} = -\frac{1}{1 - x - y}$$

   and we have $H_{xx}(1/3, 1/3) = -6 < 0$, $D(1/3, 1/3) = H_{xx}(1/3, 1/3)H_{yy}(1/3, 1/3) - H_{xy}(1/3, 1/3)^2 = 36 - 9 = 27 > 0$. This implies that the function H has a local maximum at $(1/3, 1/3)$. Hence H attains maximum value $\ln 3$ at $(1/3, 1/3)$ with $z = 1 - x - y = 1/3$.

   (c) Let $g(x, y, z) = x + y + z$. Then we are finding the maximum value of H with a constraint $g(x, y, z) = 1$. By Lagrange multiplier, we have

$$\nabla H = (H_x, H_y, H_z) = (-\ln x - 1, -\ln y - 1, -\ln z - 1) = \lambda \nabla g = \lambda(1, 1, 1) = (\lambda, \lambda, \lambda)$$

   so we have $x = y = z = 1/3$ (from the constraint $x + y + z = 1$). We have $H(1/3, 1/3, 1/3) = \ln 3$, and one can check that this is the maximum by comparing it with other values of H, for example $H(1/4, 1/4, 1/2) = \frac{3}{2} \ln 2$, which is smaller that $\ln 3$ (take exponential to compare).