Math 53 (Multivariable Calculus), Section 102 & 108
Week 8, Friday
Oct 14, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

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1. Evaluate the following double integrals.
   (a) $\int_0^1 \int_1^2 (x + e^{-y}) dxdy$
   (b) $\int_0^1 \int_0^1 \sqrt{x+y} dxdy$
   (c) $\int_0^1 \int_{3y}^3 e^{x^2} dxdy$
   (d) $\iint_D \sqrt{4 - x^2 - y^2} dxdy$, D is the disc with center the origin and radius 2.

Solution

1. (a)

$$\int_0^1\int_1^2 (x+e^{-y})dxdy = \int_0^1 \left[\frac{1}{2}x^2 + xe^{-y}\right]_1^2 dy$$
$$= \int_0^1 \left(\frac{3}{2} + e^{-y}\right) dy = \left[\frac{3}{2}y - e^{-y}\right]_0^1 = \frac{5}{2} - e^{-1}$$

(b)

$$\int_0^1\int_0^1 \sqrt{x+y}dxdy = \int_0^1 \left[\frac{2}{3}(x+y)^{\frac{3}{2}}\right]_0^1 dy$$
$$= \int_0^1 \left(\frac{2}{3}(1+y)^{\frac{3}{2}} - \frac{2}{3}y^{\frac{3}{2}}\right) dy$$
$$= \left[\frac{4}{15}(1+y)^{\frac{5}{2}} - \frac{4}{15}y^{\frac{5}{2}}\right]_0^1 = \frac{4}{15}2^{\frac{5}{4}} - \frac{8}{15}$$

(c) We'll use Fubini's theorem to interchange the order of integration.

$$\int_0^1\int_{3y}^3 e^{x^2}dxdy = \int_0^3\int_0^{x/3} e^{x^2}dydx$$
$$= \int_0^3 \frac{x}{3}e^{x^2}dx = \frac{1}{3}\left[\frac{1}{2}e^{x^2}\right]_0^3 = \frac{1}{6}(e^9-1)$$

(d) Here's an easy way. The graph of $z = \sqrt{4-x^2-y^2} \Rightarrow x^2+y^2+z^2 = 4$ is a hemisphere (upper half of a sphere) of radius 2. Hence the integral equals to the half of the volumn of the ball of radius 2, which is $\frac{1}{2}\frac{4}{3}\pi 2^3 = \frac{16}{3}\pi$.

You can also compute this directly. The bounds of integral is given by $x = -\sqrt{4-y^2}$ and $x = \sqrt{4-y^2}$. Using a substitution $x = \sqrt{4-y^2}\sin\theta$, we have

$$\iint_D \sqrt{4-x^2-y^2}dxdy = \int_{-2}^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \sqrt{4-x^2-y^2}dxdy$$
$$= \int_{-2}^2\int_{-\pi/2}^{\pi/2} \sqrt{4-y^2-(4-y^2)\sin^2\theta}\sqrt{4-y^2}\cos\theta d\theta dy$$
$$= \int_{-2}^2\int_{-\pi/2}^{\pi/2} (4-y^2)\cos^2\theta d\theta dy$$
$$= \int_{-2}^2 \frac{\pi}{2}(4-y^2)dy = \left[\frac{\pi}{2}\left(4y - \frac{1}{3}y^3\right)\right]_{-2}^2 = \frac{16\pi}{3}.$$