**Math 53 (Multivariable Calculus), Section 102 & 108**
**Week 9, Friday**
**Oct 21, 2022**
**For the other materials: seewoo5.github.io/teaching/2022Fall**

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1. Find the volumn of the solid below the plane $2x+y+z=4$ and above the disk $x^2+y^2 \leq 1$.
2. Compute $\int_0^{1/2}\int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2 dxdy$ using 1) rectangular coordinate and 2) polar coordinate.
3. A lamina occupies the part of the disk $x^2+y^2 \leq 1$ in the first quadrant. Find its center of mass if the density at any point is proportional to its distance from the $x$-axis.

**Solution**

1. Let $D = \{(x, y) : x^2 + y^2 \leq 1\}$ be a unit disk. The volumn is the same as the integral of the function $z = 4 - 2x - y$ over $D$, which is

$$\begin{aligned}\iint_D (4 - 2x - 2y)dA &= \int_0^{2\pi} \int_0^1 (4 - 2r\cos\theta - r\sin\theta) r dr d\theta \\ &= \int_0^{2\pi} \int_0^1 (4r - 2r^2\cos\theta - r^2\sin\theta) d\theta \\ &= \int_0^{2\pi} \left[2r^2 - \frac{2}{3}r^3\cos\theta - \frac{1}{3}r^3\sin\theta\right]_0^1 d\theta \\ &= \int_0^{2\pi} 2 - \frac{2}{3}\cos\theta - \frac{1}{3}\sin\theta d\theta = 4\pi\end{aligned}$$

   Note that you can also find the volumn without integration. The solid is a slice of a cylinder by a plane, and you can make a cylinder with two copies of it. The corresponding cylinder has radius 1 and height 8 (this follows from that $f(0, 0) = 4$ for $f(x, y) = 4 - 2x - y$ and we multiply by 2), so that the volumn of the original cylinder becomes $\frac{1}{2} \cdot \pi \cdot 1^2 \cdot 8 = 4\pi$.

2. If you compute the integral directly (with rectangular coordinate),

$$\begin{aligned}\int_0^{1/2} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2 dx dy &= \int_0^{1/2} \left[\frac{1}{2}x^2y^2\right]_{\sqrt{3}y}^{\sqrt{1-y^2}} dy \\ &= \int_0^{1/2} \frac{1}{2}(1 - y^2 - 3y^2)y^2 dy \\ &= \int_0^{1/2} \frac{1}{2}(y^2 - 4y^4)dy = \left[\frac{1}{6}y^3 - \frac{2}{5}y^5\right]_0^{1/2} = \frac{1}{120}.\end{aligned}$$

   To compute the integral, we first express the domain of the integration by polar coordinate. By investigating the bounds of integration, it is a sector form of radius 1 of angle $\pi/6$. In other words, it is a domain defined by $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi/6$. Hence the integral becomes

$$\begin{aligned}\int_0^{\pi/6} \int_0^1 (r\cos\theta)(r\sin\theta)^2 r dr d\theta &= \left(\int_0^1 r^4 dr\right)\left(\int_0^{\pi/6} \cos\theta\sin^2\theta d\theta\right) \\ &= \frac{1}{5}\left[\frac{\sin^3\theta}{3}\right]_0^{\pi/6} = \frac{1}{120}\end{aligned}$$

3. The density is given by $\rho(x, y) = |y|$, and the lamina can be expressed in terms of polar coordinate as $D = \{(r, \theta) : r \leq 1, 0 \leq \theta \leq \pi/2\}$, so

$$\bar{x} = \frac{\iint_D x\rho(x,y)dxdy}{\iint_D \rho(x,y)dxdy} = \frac{\int_0^{\pi/2}\int_0^1 (r\cos\theta)(r\sin\theta)rdrd\theta}{\int_0^{\pi/2}\int_0^1 (r\sin\theta)rdrd\theta} = \frac{(\int_0^1 r^3dr)(\int_0^{\pi/2}\cos\theta\sin\theta d\theta)}{(\int_0^1 r^2dr)(\int_0^{\pi/2}\sin\theta d\theta)}$$

$$= \frac{\frac{1}{4}[-\frac{1}{4}\cos 2\theta]_0^{\pi/2}}{\frac{1}{3}[-\cos\theta]_0^{\pi/2}} = \frac{3}{8}$$

$$\bar{y} = \frac{\iint_D y\rho(x,y)dxdy}{\iint_D \rho(x,y)dxdy} = \frac{\int_0^{\pi/2}\int_0^1 (r\sin\theta)(r\sin\theta)rdrd\theta}{\int_0^{\pi/2}\int_0^1 (r\sin\theta)rdrd\theta} = \frac{(\int_0^1 r^3dr)(\int_0^{\pi/2}\sin^2\theta d\theta)}{(\int_0^1 r^2dr)(\int_0^{\pi/2}\sin\theta d\theta)}$$

$$= \frac{\frac{1}{4}[\frac{\theta}{2} - \frac{\sin 2\theta}{4}]_0^{\pi/2}}{\frac{1}{3}[-\cos\theta]_0^{\pi/2}} = \frac{3\pi}{16}$$

and the center of mass is $(\bar{x}, \bar{y}) = (3/8, 3\pi/16)$.