Math 53 (Multivariable Calculus), Section 102 & 108
Week 10, Wednesday
Oct 26, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

---

1. Compute
$$\iiint_E xy\,dV$$
where E lies under the plane $z = x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$.

2. Find the volume of the solid that is enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2$.

3. Compute
$$\iiint_B x^3 + \sin(yz) + 3\,dV$$
where B is the unit ball $x^2 + y^2 + z^2 \leq 1$.

Solution

1. The region E can be expressed as $E = \{(x, y, z) : 0 \le x \le 1, 0 \le y \le \sqrt{x}, 0 \le z \le x+y\}$. Hence

$$\begin{aligned}\iiint_E xy dV &= \int_0^1 \int_0^{\sqrt{x}} \int_0^{x+y} xy dz dy dx \\ &= \int_0^1 \int_0^{\sqrt{x}} xy(x+y) dy dx \\ &= \int_0^1 \left[\frac{1}{2}x^2y^2 + \frac{1}{3}xy^3\right]_0^{\sqrt{x}} dx \\ &= \int_0^1 \frac{1}{2}x^3 + \frac{1}{3}x^{\frac{5}{2}} dx = \left[\frac{1}{8}x^4 + \frac{2}{21}x^{\frac{7}{2}}\right]_0^1 = \frac{37}{168}.\end{aligned}$$

2. With cylinderical coordinate, we can express the region as $z \ge \sqrt{x^2 + y^2} = r$ and $x^2 + y^2 + z^2 = r^2 + z^2 \le 2$. Then the projection of it to the xy-plane is a disk $r \le 1$, since $z = r$ and $r^2 + z^2 = 2$ gives $r = z = 1$. Hence the volume is

$$\begin{aligned}\text{volume} &= \int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} r dz dr d\theta \\ &= \int_0^{2\pi} \int_0^1 r(\sqrt{2-r^2} - r) dr \\ &= 2\pi \cdot \left[-\frac{1}{3}(2-r^2)^{\frac{3}{2}} - \frac{1}{3}r^3\right]_0^1 = \frac{2\pi}{3}(2\sqrt{2} - 2).\end{aligned}$$

3. Use symmetry. The domain of integration B is symmetric. Since the function $(x, y, z) \mapsto x^3$ is an odd function with respec to x (i.e. $(-x)^3 = -x^3$), we have $\iiint_B x^3 dV = 0$ since the integral on $B \cap \{x > 0\}$ and $B \cap \{x < 0\}$ cancel out. Similarly, the function $(x, y, z) \mapsto \sin(yz)$ is an odd function in y and z and we have $\iiint_B \sin(yz) dV = 0$. Hence, the integral equals to $\iiint_B 3dV$, which is the same as $3 \cdot \text{volume}(B) = 4\pi$.