**Math 53 (Multivariable Calculus), Section 102 & 108**
**Week 10, Friday**
**Oct 28, 2022**
**For the other materials: seewoo5.github.io/teaching/2022Fall**

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1. Let $0 \leq a < b \leq 1$. Find the area of the part of the unit sphere $x^2 + y^2 + z^2 = 1$ that lies above the plane $z = a$ and below the plane $z = b$. Check that the result only depends on the value $(b - a)$.

2. Evaluate
$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{1-\sqrt{1-x^2-y^2}}^{1+\sqrt{1-x^2-y^2}} (x^2 + y^2 + z^2)^{1/2} dz dy dx.$$
(Hint: Change to the spherical coordinate.)

**Solution**

1. The (upper) sphere can be expressed as a graph of $z = f(x, y) = \sqrt{1 - x^2 - y^2}$. Its first derivatives are

$$f_x = \frac{-x}{\sqrt{1 - x^2 - y^2}}, \quad f_y = \frac{-y}{\sqrt{1 - x^2 - y^2}}$$

To found the area, we first find the domain of the integration for the surface area formula. The intersections of the sphere with planes $z = a$ and $z = b$ are circles, and their projections onto the $xy$-plane encloses an annulus $D : r_1 \leq \sqrt{x^2 + y^2} \leq r_2$, where $r_1 = \sqrt{1 - b^2}$ and $r_2 = \sqrt{1 - a^2}$. Hence, the surface area is

$$\begin{aligned}
\iint_D \sqrt{1 + f_x^2 + f_y^2}\,dA &= \iint_D \sqrt{1 + \frac{x^2}{1 - x^2 - y^2} + \frac{y^2}{1 - x^2 - y^2}}\,dA \\
&= \int_0^{2\pi} \int_{r_1}^{r_2} \frac{1}{\sqrt{1 - r^2}} r\,dr\,d\theta \qquad \text{(polar coordinate)} \\
&= \int_0^{2\pi} \int_{r_1^2}^{r_2^2} (1 - u)^{-1/2} \frac{1}{2} du\,d\theta \qquad (u = r^2) \\
&= 2\pi \Big[ -(1 - u)^{1/2} \Big]_{r_1^2}^{r_2^2} = 2\pi(\sqrt{1 - r_1^2} - \sqrt{1 - r_2^2}) = 2\pi(b - a)
\end{aligned}$$

and it depends only on $(b - a)$.

2. The bounds for $z$ tells us that the domain of integration is given by

$$1 - \sqrt{1 - x^2 - y^2} \leq z \leq 1 + \sqrt{1 - x^2 - y^2} \Leftrightarrow (z - 1)^2 \leq 1 - x^2 - y^2$$

hence the domain is a ball $x^2 + y^2 + (z - 1)^2 \leq 1$. If you plug $x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi$, it gives $\rho \leq 2\cos\phi$. Also, $0 \leq \phi \leq \pi/2$ since the sphere is above the $xy$-plane and tangent to it at the origin. Hence, the integral equals to

$$\begin{aligned}
\int_0^{2\pi} \int_0^{\pi/2} \int_0^{2\cos\phi} \rho \cdot \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta &= \int_0^{2\pi} \int_0^{\pi/2} \left[ \frac{1}{4}\rho^4 \right]_0^{2\cos\phi} \sin\phi\,d\phi\,d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/2} 4\cos^4\phi \sin\phi\,d\phi\,d\theta \\
&= \int_0^{2\pi} \int_1^0 4u^4(-du)\,d\theta \qquad (u = \cos\theta) \\
&= \int_0^{2\pi} \left[ \frac{4}{5}u^5 \right]_0^1 d\theta = \frac{8\pi}{5}.
\end{aligned}$$