**Math 53 (Multivariable Calculus), Section 102 & 108**
**Week 11, Monday**
**Oct 31, 2022**
**For the other materials: seewoo5.github.io/teaching/2022Fall**

---

1. Evaluate
$$\iint_R \frac{(x-y)^2}{x+y-2} dA$$
where $R$ is given by the inequality $|x| + |y| \leq 1$. Use the transformation $u = x + y$ and $v = x - y$.

2. Find the volume of the region bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and and the coordinate planes. Use the transformation $x = u^2, y = v^2, z = w^2$.

**Solution**

1. The region $R$ is enclosed by the lines $x+y=1$, $x+y=-1$, $y=x+1$, and $y=x-1$. With the transformation $u=x+y$ and $v=x-y$, these correspond to $u=1$, $u=-1$, $v=-1$, and $v=1$. $x, y$ can be written as $x=\frac{u+v}{2}$ and $y=\frac{u-v}{2}$, so the Jacobian of the transformation is
$$J(u,v)=\begin{vmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{vmatrix}=-1/2.$$
Hence
$$\iint_R \frac{(x-y)^2}{x+y-2}dA=\int_{-1}^{1}\int_{-1}^{1}\frac{v^2}{u-2}\left|\frac{1}{2}\right|dudv=\frac{1}{2}\left(\int_{-1}^{1}\frac{1}{u-2}du\right)\left(\int_{-1}^{1}v^2dv\right)=-\frac{\ln 3}{3}.$$

2. With the transformation, transformed region $E$ can be written as $u, v, w \geq 0$ and $u+v+w \leq 1$. Also, the Jacobian of the transformation becomes
$$J(u,v,w)=\begin{vmatrix}2u & 0 & 0\\ 0 & 2v & 0\\ 0 & 0 & 2w\end{vmatrix}=8uvw.$$
Hence the volume equals to
$$\begin{aligned}\iiint_E (8uvw)dudvdw &= \int_0^1\int_0^w\int_0^{1-v-w} 8uvwdudvdw\\ &= \int_0^1\int_0^w 4(1-v-w)^2vwdvdw\\ &= \int_0^1\int_0^w 4((1-w)^2-2(1-w)v+v^2)vwdvdw\\ &= \int_0^1 \left[2(1-w)^2wv^2-\frac{8}{3}(1-w)wv^3+wv^4\right]_0^w dw\\ &= \int_0^1 2(1-w)^2w^3-\frac{8}{3}(1-w)w^4+w^5dw\\ &= \int_0^1 2w^3-\frac{20}{3}w^4+\frac{17}{3}w^5dw=\frac{1}{9}.\end{aligned}$$