Math 53 (Multivariable Calculus), Section 102 & 108
Week 11, Wednesday
Nov 2, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

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1. Find the gradient vector field ∇f of f and sketch it.
   (a) $f(x, y) = \frac{1}{2}(x^2 - y^2)$
   (b) $f(x, y) = \ln \sqrt{x^2 + y^2}$

2. Compute the following line integrals.
   (a) $\int_C y ds$, where $C : (x(t), y(t)) = (t^2, 2t)$, $0 \leq t \leq 3$.
   (b) $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ is a vector field

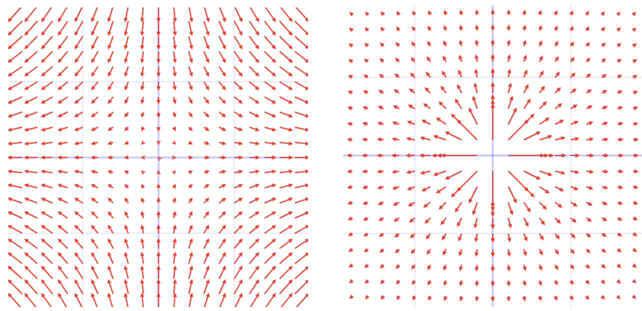
$$\mathbf{F}(x, y) = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)$$

   and C is a circle $x^2 + y^2 = a^2$ with parametrization given by $\mathbf{r}(t) = (a \cos t, a \sin t)$, $0 \leq t \leq 2\pi$.

Solution

1. (a) $\nabla f = (x, -y)$.

   (b) $\nabla f = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$. The length of ∇f is inversely proportional to the distance between (x, y) and the origin.



(a) (b)

2. We have $x'(t) = 2t$ and $y'(t) = 2$. Hence

$$\int_C y ds = \int_0^3 2t\sqrt{(2t)^2 + 2^2} dt = \int_0^3 4t\sqrt{t^2+1} dt = \left[\frac{4}{3}(t^2+1)^{3/2}\right]_0^3 = \frac{4}{3}(10^{3/2} - 1)$$

3. $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left(-\frac{a\sin t}{a^2}, \frac{a\cos t}{a^2}\right) \cdot (-a\sin t, a\cos t) dt = \int_0^{2\pi} dt = 2\pi$$

   Note that the value of the line integral only depends on *how many times the curve C wind the origin*. For any curve C that wind the origin for n times counter-clockwise, the integral becomes $2\pi n$.