Math 53 (Multivariable Calculus), Section 102 & 108
Week 11, Friday
Nov 4, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

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1. Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

   (a) $\mathbf{F}(x, y) = y^2 e^{xy}\mathbf{i} + (1 + xy)e^{xy}\mathbf{j}$

   (b) $\mathbf{F}(x, y) = \sin(x + y)\mathbf{i} + \cos(x - y)\mathbf{j}$

2. (a) Let
$$\mathbf{F}(x, y) = (\cos(xy) - xy\sin(xy))\mathbf{i} + (-x^2\sin(xy))\mathbf{j}.$$
   Find $f(x, y)$ such that $\mathbf{F} = \nabla f$.

   (b) Using (a), compute
$$\int_C (\cos(xy) - xy\sin(xy) + x)dx + (-x^2\sin(xy) + y)dy$$
   where C is an arc from $(-1, 0)$ to $(1, 0)$, along the unit circle $x^2 + y^2 = 1$. (Hint: decompose the integral into sum of two line integrals.)

Solution

1. Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$.

   (a) We have $P_y = 2ye^{xy} + xy^2e^{xy} = Q_x$ and since P, Q are vector field on whole $\mathbb{R}^2$ (which is simply connected), it is conservative. Since we have $f_x(x, y) = y^2e^{xy}$, we have $f(x, y) = ye^{xy} + g(y)$ for some function $g(y)$ on y, and since $f_y(x, y) = (1 + xy)e^{xy} + g'(y) = Q(x, y)$, we can simply take $g = 0$ and get $f(x, y) = ye^{xy}$.

   (b) Since $P_y = \cos(x + y)$ and $Q_x = -\sin(x - y)$, they are difference and $\mathbf{F}$ is not conservative.

2. (a) Since $f_y(x, y) = -x^2 \sin(xy)$, $f(x, y) = x\cos(xy) + g(y)$. Now $f_x(x, y) = \cos(xy) - xy\sin(xy) + g'(y)$ so we can take $g = 0$ and $f(x, y) = x\cos(xy)$.

   (b) Let $\mathbf{G}(x, y) = x\mathbf{i} + y\mathbf{j}$. Then then integral is $\int_C(\mathbf{F} + \mathbf{G}) \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{G} \cdot d\mathbf{r}$. Since $\mathbf{F}$ is conservative with $\mathbf{F} = \nabla f$, we have

   $$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 0) - f(-1, 0) = 2.$$

   Also, we can directly compute $\int_C \mathbf{G} \cdot d\mathbf{r} = \int_C x dx + y dy$. If we choose a parametrization of C as $\mathbf{r}(t) = (-\cos t, \sin t)$ for $0 \le t \le \pi$, we have

   $$\int_C \mathbf{G} \cdot d\mathbf{r} = \int_0^\pi (-\cos t, \sin t) \cdot (\sin t, \cos t) dt = 0.$$

   Hence the original line integral equals to $2 + 0 = 2$.

   In fact, the vector field $\mathbf{F} + \mathbf{G}$ is also conservative, and $\mathbf{F} + \mathbf{G} = \nabla(x\cos(xy) + (1/2)(x^2 + y^2))$. You can use this to deduce the same result, without decomposing it.