Math 53 (Multivariable Calculus), Section 102 & 108
Week 12, Monday
Nov 7, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

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1. Let $\mathbf{F}(x, y, z) = y^2 z^3 \mathbf{i} + 2xyz^3 \mathbf{j} + 3xy^2 z^2 \mathbf{k}$. If C is a line segment from $(0,0,0)$ to $(1,1,1)$, find $\int_C \mathbf{F} \cdot d\mathbf{r}$.

2. Let
$$\mathbf{F}(x, y) = (2xye^{x^2 y} + 1)\mathbf{i} + (x^2 e^{x^2 y} + 1)\mathbf{j}.$$

   (a) Is $\mathbf{F}$ conservative? If it is, find $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla f = \mathbf{F}$.

   (b) Let C be a curve parametrized by $\mathbf{r}(t) = (\cos t, \sin t)$ for $0 \leq t \leq \pi/2$. Find
$$\int_C (2xye^{x^2 y} + 1 - y)dx + (x^2 e^{x^2 y} + 1 + x)dy.$$
   (Hint: express the vector field in the integral as $\mathbf{F} + \mathbf{G}$ for some vector field $\mathbf{G}$.)

Solution

1. The vector field $\mathbf{F}$ is conservative with $\nabla f = \mathbf{F}$ for $f(x, y, z) = xy^2z^3$. Such f can be found as follows: once you know that $\mathbf{F}$ is conservative (for example, by showing its curl is identically zero and the vector field is defined on whole $\mathbb{R}^3$), we can set $\nabla f = \mathbf{F}$ and try to find f. From $f_x = y^2z^3$, we have $f(x, y, z) = xy^2z^3 + g(y, z)$ for some g, and by comparing f_y and f_z with components of $\mathbf{F}$, we can find that $g(y, z)$ is a constant function and can take $g = 0$ and $f(x, y, z) = xy^2z^3$. Hence the line integral becomes $f(1, 1, 1) - f(0, 0, 0) = 1$.

   It is also possible to compute the line integral directly using the parametrization $\mathbf{r}(t) = (t, t, t)$, which gives the same answer.

2. (a) Since $P_y = 2xe^{x^2y} + 2x^3ye^{x^2y} = Q_x$ and defined on $\mathbb{R}$ (which is simply connected), it is conservative. If we let $\nabla f = \mathbf{F}$, then $f_x(x, y) = 2xye^{x^2y} + 1$ and so $f(x, y) = e^{x^2y} + x + g(y)$ for some $g : \mathbb{R} \to \mathbb{R}$. Now $f_y(x, y) = x^2e^{x^2y} + g'(y) = x^2e^{x^2y} + 1$, so we have $g'(y) = 1$. We can take $g(y) = y$ and this gives $f(x, y) = e^{x^2y} + x + y$.

   (b) The line integral equals to $\int_C (\mathbf{F} + \mathbf{G}) \cdot d\mathbf{r}$, where $\mathbf{G}(x, y) = -y\mathbf{i} + x\mathbf{j}$. Then we can decompose it as $\int_C \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{G} \cdot d\mathbf{r}$ and compute each integral seperately. From (a), we get $\int_C \mathbf{F} \cdot d\mathbf{r} = f(0, 1) - f(1, 0) = 2 - 2 = 0$. The other line integral can be directly computed as

$$\int_C -ydx + xdy = \int_0^{\pi/2} (-\sin t, \cos t) \cdot (-\sin t, \cos t)dt = \int_0^{\pi/2} 1dt = \frac{\pi}{2}.$$

   Hence the anwer is $0 + \pi/2 = \pi/2$.