Math 53 (Multivariable Calculus), Section 102 & 108
Week 12, Wednesday
Nov 9, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

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1. Use Green's theorem to evaluate the line integral along given positively oriented curve.
   (a) $\int_C y^4 dx + 2xy^3 dy$, C is the ellipse $x^2 + 2y^2 = 2$.
   (b) $\int_C (e^{-x} + y^2)dx + (e^{-y} + x^2)dy$, C consists of the arc of the curve $y = \cos x$ from $(-\pi/2, 0)$ to $(\pi/2, 0)$ and the line segment from $(\pi/2, 0)$ to $(-\pi/2, 0)$.
2. Let D be a region bounded by a simple closed path C. If D has area 3, what is the value of $\int_C (\sin x + 7y)dx + (e^y + 10x)dy$?

Solution

1. (a) By Green's theorem, we have

$$\begin{aligned}\int_C y^4dx + 2xy^3dy &= \iint_D (2y^3 - 4y^3)dxdy \\ &= \iint_D (-2y^3)dxdy\end{aligned}$$

Once you notice that the domain D (interior of ellipse) is symmetric and $-2y^3$ is an odd function, the integration would be zero.

(b) By Green's theorem, we have

$$\begin{aligned}\int_C (e^{-x} + y^2)dx + (e^{-y} + x^2)dy &= \iint_D (2x - 2y)dxdy \\ &= \int_{-\pi/2}^{\pi/2}\int_0^{\cos x}(2x - 2y)dydx \\ &= \int_{-\pi/2}^{\pi/2}[2xy - y^2]_0^{\cos x}dx \\ &= \int_{-\pi/2}^{\pi/2}(2x\cos x - \cos^2 x)dx\end{aligned}$$

The integration of $2x\cos x$ can be done by using integration by parts:

$$\int_{-\pi/2}^{\pi/2} 2x\cos x dx = [2x\sin x]_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} 2\sin x dx = -[-2\cos x]_{-\pi/2}^{\pi/2} = 0$$

(if you noticed that the function $2x\cos x$ is an odd funciton, then you will automatically get zero.) The second term can be done using $\cos^2 x = (\cos 2x + 1)/2$, so that

$$\int_{-\pi/2}^{\pi/2}\cos^2 x dx = \int_{-\pi/2}^{\pi/2}\frac{1+\cos 2x}{2}dx = \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_{-\pi/2}^{\pi/2} = \frac{\pi}{2}.$$

Hence the answer is $0 - \pi/2 = -\pi/2$.

2. By Green's theorem, we have

$$\int_C (\sin x + 7y)dx + (e^y + 10x)dy = \iint_D (10 - 7)dxdy = 3\iint_D dxdy = 3 \cdot \text{area}(D) = 9.$$