What is... Multivariable Calculus?

Seewoo Lee

1 What you (may) have learned

If you are a Berkeley student and took Math 1A and 1B course before, then you may have learned the following stuffs:

- *Single-variable* functions
- *Single-variable* limit
- *Single-variable* differentiation
- *Single-variable* integration
- *Single-variable* differential equation

As you noticed, I'm emphasizing *single-variable*. You have learned about functions defined on the set of real numbers (or subset of it), and their continuity, differentiation, and integration. Also, there are tons of applications of differentiations and integration of functions in Physics, Engineering, Chemistry, etc.

Limit is basically about the behaviour of a function near a specific point. We use the notation

$$\lim_{x\to a} f(x)$$

for the limit of the function $f(x)$ as x approaches to a. It may or may not exists, and also could be different from the value $f(a)$. We use *epsilon-delta definition* of limit to define the limit rigorously, although we intuitively know what is the limit is. (Mathematicians like to define something rigorously, even if it looks trivial for us.) We call that a function $f(x)$ is *continuous at* $x = a$ if the limit equals to $f(a)$.

Differentiation measures the change of the function near a specific point. It is defined as a limit of average rate of change, which is

$$\lim_{x\to a} = \frac{f(x) - f(a)}{x - a} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

and this gives the slope of a tangent line of $f(x)$ at $x = a$. It may or may not exists (as limit does), and we call that the function $f(x)$ is *differentiable at* $x = a$ if the limit exists, and denote it by $f'(a)$. Intuitively, it means that the function $f(x)$ can be well-approximated by a tangent line at the point $x = a$. You may have learned formula on differentiations of special functions like polynomials, rational functions, trigonometric functions, exponential and logarithms, etc. Also, using the Leibniz rule and the chain rule help us to find differentiation of complicated functions easily.

- Leibniz rule: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
- Chain rule: $f(g(x))' = f'(g(x))g'(x)$

For example, the differentiation of $f(x) = x\sin(x^2)$ can be computed as

$$\begin{aligned}
(x\sin(x^2))' &= (x)'\sin(x^2) + x(\sin(x^2))' && \text{(Leibniz rule)}\\
&= \sin(x^2) + x(\sin(x^2))'\\
&= \sin(x^2) + x\cdot\cos(x^2)\cdot(x^2)' && (\text{Chain rule}, \sin(x)' = \cos(x))\\
&= \sin(x^2) + x\cdot\cos(x^2)\cdot 2x\\
&= \sin(x^2) + 2x^2\cos(x^2)
\end{aligned}$$

Also, there's a theorem that if a function $f(x)$ has a local maximum or minimum at $x = a$, and it is differentiable at $x = a$, then $f'(a) = 0$. This let us to find the *candidates* of points that $f(x)$ attains a (local) maximum or minimum. To determine if the point is actually (local) maximum or minimum, observing the sign of the second order derivative $f''(a)$ would be helpful.

Integration is about finding an area under a graph of a certain function. We use the following notation

$$\int_a^b f(x)dx$$

for the area under the graph of the function $f(x)$ with $0 \le x \le b$. For example, the area under the function $f(x) = x$ with $0 \le x \le 2$ is a right triangle with both base and height has lenght 2, so that the area would be $\frac{1}{2} \times 2 \times 2 = 2$. However, you may need to find the area under more complicated functions (like polynomials, trigonometric functions, ...), and we need a new approach for these. The idea of integration is to divide the region under the function into rectangles with tiny widths, and approximate the area as a sum of the area of these rectangles, and take a limit of width $\to 0$ to get the area we want. The most important theorem is the *Fundamental Theorem of Calculus*, which relates the (definite) integration with antiderivatives (indefinite integraion). In summary, if you know the *antiderivative* $F(x)$ of a function $f(x)$, i.e. the function $F(x)$ with $F'(x) = f(x)$ on $a \le x \le b$, then the integration will be

$$\int_a^b f(x)dx = F(b) - F(a).$$

Also, there are several techniques like *Integration by Parts* and *Change of Variables* to compute integrals more easily. (These techniques essentially corresponds to the Leibniz rule and chain rule for differentiation.) In general, it is much easier to find the derivative of a function than integration (some of the functions like $1/\log(x)$ even don't have antiderivatives of closed form.)

At last, you learned about *differential equations*, which are equations on functions that consist of derivatives of them. For example, it may asks you to find a function $y = f(x)$ that satisfies $y'' + y = 0$. It is not hard to check that the functions of the form $A\sin(x) + B\cos(x)$ for any constants A, B satisfy the equation. However, it is hard in general to find the solution(s) of given differential equation. Some of (actually, most of) the differential equations have no closed-form solutions, and we need a computer to find the approximations of them numerically.

2 What you are going to learn

What you are going to learn in Math 53 will be the multivariable version of the above topics that you have learned from Math 1A and Math 1B. So basically, you are going to learn about

- Parametric equations and curves
- Vectors and vector functions
- *Multivariable* functions
- *Multivariable* limit
- *Multivariable* differentiation
- *Multivariable* integration
- Green's Theorem, Stokes' Theorem, Divervence Theorem
- ~~*Multivariable* differential equation~~

The most important topic in multivariable calculus is about limit, differentiation, and integration of *multivariable* functions, which are functions on several variables. (We will only going to study functions on 2 or 3 variables in this course.) For example, $f(x, y) = x^2 + 3xy$ is a function on two variables x and y. In case of limit, the situation becomes much harder than the limt of single-variable functions. For the single-variable case, there are essentially (at most) *two* directions that the input x can approach to a point a: from the above ($x > a$) and below ($x < a$). However, for the multi-variable functions, there are *infinitely many* ways to approach a given point (a, b) (see below). Like single-variable case, the rigorous definition uses epsilon-delta.

Let's consider a two-variable function $f(x, y)$. If we fix y as $y = b$ then we can regard it as a single-variable function $x \mapsto f(x, b)$, and we can say differentiation of it. We call it as a *partial derivative of* f *with respect to* x *at* (a, b), which is just

$$\lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

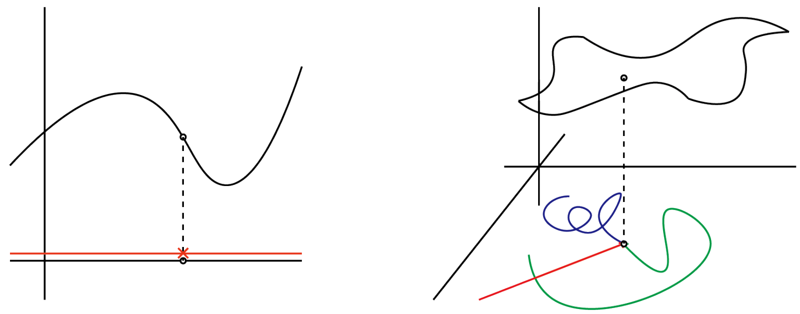

Figure 1: Limit of a s1ingle-variable function (left) and a 2-variable function (right).

and we denote it as $f_x(a,b)$ (if exists). As you expect, we denote the partial derivative with respect to y as $f_y(a,b)$, and we can differentiate more and get things like $f_{xx}, f_{xy}, f_{xyyx}, \ldots$ if you want (and if you can). Also, it is even possible to define *directional derivative* of a function in the direction of any given (unit) vector $\mathbf{u} = \langle a, b\rangle$ at a point (x_0, y_0), which is

$$D_{\mathbf{u}}f(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}.$$

You can check that the directional derivatives in the direction of $\langle 1,0\rangle$ and $\langle 0,1\rangle$ are just f_x and f_y. Also the directional derivative $D_{\mathbf{u}}f(x,y)$ can be expressed as an inner product

$$D_{\mathbf{u}}f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle \cdot \mathbf{u} = \nabla f \cdot \mathbf{u}$$

where $\nabla f = \langle f_x(x,y), f_y(x,y)\rangle$ is a special and important vector function called the *gradient of* f. It gives the direction of fastest increase of f.

Finding (local) maximum and minimum values of multivariable functions also uses differentiation. Like a single-variable case, the partial derivatives f_x and f_y should be zero at the local maximum and minimum. (We call such a point as a *critical point*.) To determine if the function attins local maximum or minimum (or neither), there's also a second derivative test for multi-variable functions. Also, it is possible to find maxiumum or minimum of a multivariable function $f(x,y)$, subject to a constraint by another multivariable function as $g(x,y) = k$, using *Langrange Multiplier*.

Integration of multi-variable function is to find the *volume* of an area under a graph of a multi-variable function. For example, we may want to find a volume of an area under the graph of function $f(x,y) = x^2 + 3xy$ over a region $D = \{(x,y)|x^2+y^2 \le 1\}$ enclosed by a circle. We call it as the *double integral of* f *over* D, and denote by

$$\iint_D f(x,y)dA.$$

You will learn various techniques and theorems to compute double integrals, such as Fubini's theorem and change of variables. Sometimes it is easier to compute double integrals using *polar coordinates* instead of the rectangular coordinates. Also, it is possible to express the *surface area* of a graph of a function as a certain double integral, and compute it, which is very similar to the arc length formula for single-variable functions.

We can do similar things for 3-variable functions (actually for many-variable functions, but we are not going to deal with it in this course). We can define *triple integral* of a function $f(x, y, z)$ over some region, and use it to compute volume or mass enclosed by a level set of a function. (As you expected, we use three integral symbols for triple integral.)

The most important part of multivariable calculus is Green's theorem, Stokes' theorem, and Divergence theorem. Firstly, we define *line integral* of a function or *vector field* over a curve as an integral of the function / vector field along a curve. Green's theorem relates a *line integral* around a plane curve C and a double integral over the plane region D bounded by C. Hence we can translate difficult line integrals into more simple double integrals, or vice versa. Stokes' theorem is a 3-dimensional version of the Green's theorem, which relates *line* integral of a 3-dimensional vector field $\mathbf{F}$ over a curve C with a *surface* integral of another vector field called *curl* of $\mathbf{F}$, denoted as $\operatorname{curl}\mathbf{F}$, over a surface S whose boundary is C. At last, the divergence theorem gives an equality between a surface integral over a boundary surface S of a region E and the triple integral of the *divergence* of $\mathbf{F}$, $\operatorname{div}\mathbf{F}$, over E. The following figure (Figure 2) gives intuitions for curl and divergence, along with the conceptual proof of these theorems.[1]

Differential equations for multivariable functions are often called *Partial Differential Equations*, and we are not going to learn deeply on the topic. As in the case of single-variable, it is easy to check that a given function satisfies a certain partial differential equation, while finding solutions of a given equation is extremely hard in general. For example, *Navier-Stokes equations* are certain class of partial differential equations that describe the motion of fluid, and it is one of the seven millenial problems in mathematics.

[1] If you have a chance to learn *differential geometry*, then the functions like gradient, curl, and divergence can be generalized into higher dimensional objects (more than 3) called *differential forms*. In this sense, the three theorems mentioned above are just specific cases of a single theorem from Stokes.

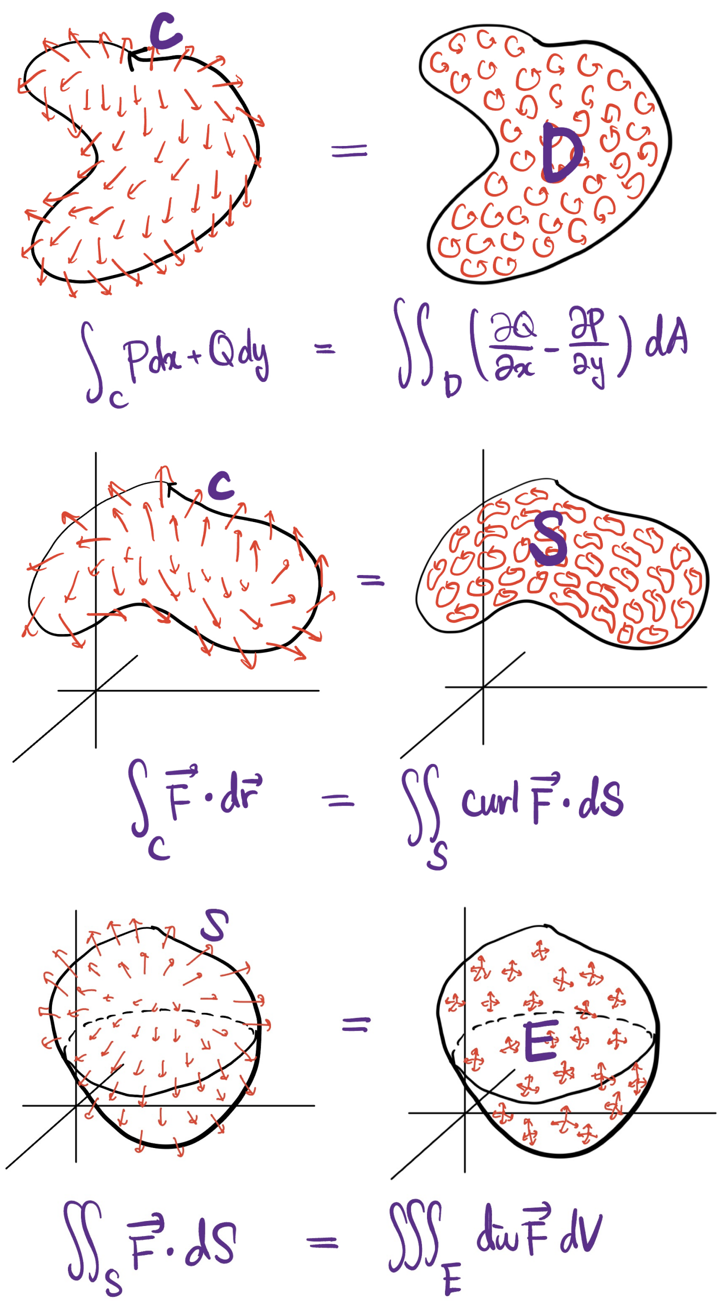

Figure 2: Intuitive explanations for Green's theorem, Stokes' theorem, and Divergence theorem. curl $\mathbf{F}$ can be thought as *local rotation of* $\mathbf{F}$ *at a point*, and div $\mathbf{F}$ can be thought as *local divergence of* $\mathbf{F}$.